Francis. H. Lee. 1851.

No 14 Chestnut St.

Salem

Mass.

Uti sed non Abuti:

Charles. J. Lee. 1853

Salem Mass

FIRST LESSONS

IN

ALGEBRA,

BEING AN

EASY INTRODUCTION TO THAT SCIENCE:

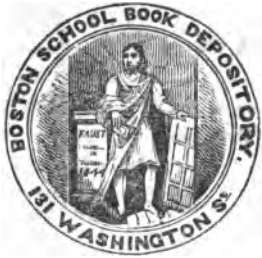
DESIGNED FOR THE USE OF

ACADEMIES AND COMMON SCHOOLS.

BY EBENEZER BAILEY,

PRINCIPAL OF THE YOUNG LADIES' HIGH SCHOOL, BOSTON; AUTHOR OF
"YOUNG LADIES' CLASS BOOK," ETC.

THIRTY-FIRST IMPROVED STEREOTYPE EDITION.



BOSTON:
JENKS & PALMER.
1847.

AT a meeting of the School Committee of the City of Boston, March 11, 1834,

Voted, That "BAILEY'S ALGEBRA" be used in the Writing Schools in which Algebra is allowed to be taught.

Attest,

S. F. McCLEARY, *Secretary*

PREFACE.

This treatise is especially intended for the use of beginners. I have long wished that Algebra might be introduced into common schools, as a standard branch of education; and there seems to be no good reason why the study of this most interesting and useful science should be confined to the higher seminaries of learning. The upper classes, at least, in common schools, might be profitably instructed in its elements, without neglecting any of those branches to which they usually attend.

This work pretends to no original investigations, no new discoveries. My labor has been the very humble one of selecting such materials as belong to the elements of Algebra, and of arranging them in such a manner as may render the introduction to the science easy. If there be any peculiarity in this work, it is its simplicity. I have endeavored to make it as plain and

intelligible as possible. There is little danger that the student will find the beginning of any art or science too easy; and, in Algebra, he is required to learn a peculiar language, to determine new principles, and to accustom himself to an abstract mode of reasoning, with which he has been little acquainted. Let the explanations, therefore, be as full and diffuse as they may, he will still find difficulties enough to exercise his mind. I have aimed to prepare a work, which any boy of twelve years, who is thoroughly acquainted with the fundamental rules of Arithmetic, can understand, even without the aid of a teacher.

The following are the leading principles which I have observed, in preparing this treatise:—

To introduce only such parts of the science, as properly belong to an elementary work;

To adhere strictly to a methodical arrangement, that can be easily understood and remembered;

Never to anticipate principles, so as to make a clear understanding of the subject under consideration, depend upon some explanation which is to follow;

To introduce every new principle distinctly by itself, that the learner may encounter but one difficulty at a time;

To deduce the rules, generally, from practical exercises, and to state them distinctly and in form;

To give a great variety of questions for practice under each rule;

To solve or fully explain all questions which involve a new principle, or the new application of a principle already explained;

To show the reason of every step, without perplexing the learner with abstruse demonstrations;

To illustrate the nature of algebraic calculations, and their correctness, by a frequent reference to numbers;

And, finally, to advance from simple to difficult problems in such a manner as may fully exercise the powers of the learner without discouraging him.

As this little book professes to be merely an introduction to more full and scientifical treatises upon Algebra, it was not my original design to extend it beyond Equations of the First Degree. The subsequent Chapters, on Evolution and Equations of the Second Degree, have been added with a particular reference to schools for young ladies. It is presumed that the work, in its present form, contains as much of Algebra as this class of learners will, in general, find time to study.

E. BAILEY.

Boston, July, 1833.

ADVERTISEMENT

TO THE STEREOTYPE EDITION.

The favor with which this treatise has been received by the public,—as manifested by the sale of the first edition, consisting of two thousand copies, in a few months, and by commendatory notices from teachers and others in almost every section of the country,—has induced the publishers to stereotype the work, and thus put it into a permanent form.

In preparing this edition, the author has made such additions and alterations as experience has suggested; and he trusts that he has rendered The First Lessons in Algebra still more worthy of public favor. The arrangement of several parts of the work has been changed; some of the Chapters have been nearly rewritten, especially those on Powers and Evolution; the errors of the first edition have been carefully corrected; and many questions for practice have been added. It has, also, been thought advisable to omit the answers to the questions. These are given in a Key, published separately, together with solutions of all the difficult problems.

Boston, January, 1834.

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FIRST LESSONS IN ALGEBRA

CHAPTER I.

INTRODUCTION.

SECTION I.

Letters and Coefficients.

1. A BOY bought a peach and a melon for 12 cents and the melon cost three times as much as the peach What was the price of each?

Let the letter x represent the number of cents the boy gave for the peach; then, as he gave one x for the peach, whatever the value of x may be, and as the melon cost three times as much, he must have given three x's for that; and, of course, he gave one x and three x's, that is, four x's for both. But, by the question, he gave 12 cents for both; therefore, the four x's must be equal in value to the 12 cents. But, if four x's are equal to 12 cents, one x must be equal to one fourth part of 12 cents, or 3 cents, which was the price of the peach; and if one x be

equal to 3 cents, three *x's* must be equal to three times 3 cents, or 9 cents, the price of the melon.

It will be observed, that, in this operation, the *answer* of the question, the thing *unknown*, is assumed and represented by the letter x, which is therefore called the *unknown quantity*. Any other letter, mark or character, may be used with equal propriety, provided always that its value be indefinite. This will be evident if the word *share* or *part* be substituted for the letter x, in the above operation. It is usual, however, to *represent unknown quantities by the last letters of the alphabet*, as x, y, z.

It is sometimes necessary to express quantities, whose values either are, or are supposed to be, determined, by letters. *These are called* known quantities, *and are usually represented by the first letters of the alphabet*, as a, b, c.

2. John is four times as old as James, and the sum of their ages is 20 years. What is the age of each?

Let x represent the age of James; then, as John is four times as old, four *x's* will represent his age; and their joint ages must be one x and four *x's*, that is, five *x's*. But the sum of their ages is 20 years, by the question; therefore, five *x's* must be equal to 20 years, and one x to one fifth part of 20, namely, 4 years, which is the age of James; and, if one x be 4 years, four *x's* must be four times 4, or 16 years, the age of John.

Instead of writing *one x, three x's, four x's, five x's*, &c., as in these examples, we use the expressions x, $3\ x$, $4\ x$, $5\ x$, &c. *The numbers placed before the*

letters, as 3, 4, 5, *are called their coefficients.* When no number is placed before a letter, as x, its coefficient is always understood to be 1.

In the last example, the algebraic expression for John's age was $4\,x$, and the value of x was found to be four years. To find the age of John in years, this value of x was multiplied by 4, its coefficient. *Any quantity is always supposed to be multiplied by its coefficient.* Thus, if the value of x be 6, $3\,x$ will be 3 times 6, or 18; and if the value of x be 10, then $3\,x$ will be 3 times 10, or 30; and $7\,x$ will be 70, $9\,x$ will be 90, and so on.

It is often convenient, in algebraic calculations, to use a *letter* for a coefficient, instead of a *number*, as $m\,x$, where m is regarded as the coefficient of x; thus, if m be 3, and x be 5, $m\,x$ will be 3 times 5, or 15.

3. A leaves Boston, and walks three miles an hour, and B leaves Newburyport, at the same time, and walks 5 miles an hour. In how many hours will they meet, the places being 32 miles apart?

In this question, the thing required is, in how many hours A and B will meet; that is, how many hours they will travel. Let it be assumed that they will meet in x hours. Then if A walk 3 miles in 1 hour, in x hours he will walk x times 3 miles, that is, $x\,3$ or $3\,x$ miles; and if B walk 5 miles in 1 hour, in x hours he will walk $x\,5$ or $5\,x$ miles; and they will both walk $3\,x$ and $5\,x$, or $8\,x$ miles, which is the whole distance. But the distance given in the question is 32 miles; therefore, $8\,x$ miles must be the same as 32 miles; or, to use a general expression, $8\,x$ is equal in

value to 32. And if $8x$ be equal to 32, x must be one eighth part of 32, or 4. They will meet in 4 hours.

The expressions $x3$ and $3x$, used in this operation, mean the same thing; for *any two or more quantities are supposed to be multiplied together, when they are not separated;* and, of course, it is of no consequence which is placed first. Thus, if the value of x be 4, as in the question, $3x$ is 3 times 4, or 12; and $x3$ is 4 times 3, or 12. But it is more convenient to *place the number before the letter*, which is always done.

4. A farmer sold a calf, a sheep and a cow, for 36 dollars; for the sheep he received twice as much as for the calf, and for the cow three times as much as for both the calf and the sheep. What was the price of each?

Let x represent the price of the calf; then $2x$ will be the price of the sheep; x and $2x$, or $3x$, will be the price of the two, and three times $3x$, or $9x$, will be the price of the cow. The three animals were, therefore, sold for x, and $2x$, and $9x$, that is, for $12x$. But, according to the question, they were sold for 36 dollars; $12x$ must, therefore, be equal to 36 dollars, and the value of x must be one twelfth part of 36, namely, 3 dollars, which is the price of the calf: if the value of x be 3 dollars, $2x$ is twice 3, or 6 dollars, the price of the sheep; and $9x$ is 9 times 3, or 27 dollars, the price of the cow.

5. A gentleman gave a purse, containing a certain sum of money, to his three children, to be divided among them in such a manner, that Mary should have

twice as much as Ellen, and John as much as both his sisters. What was the share of each?

As the sum contained in the purse is not named, we will call it a.

Let x denote Ellen's share; then Mary's share is twice as much, or $2x$; and John's, x and $2x$, that is, $3x$; and the sum of their shares is x and $2x$ and $3x$, or $6x$, which must be equal to a, the sum to be divided, whatever the value of a may be. And if $6x$ is equal to a, x is equal to one sixth part of a, which is Ellen's share.

If the purse contained 18 dollars, Ellen's share was 3, Mary's 6, and John's 9 dollars.

If the purse contained 24 dollars, Ellen's share was 4, Mary's 8, and John's 12 dollars.

In this manner the share of each may be determined, whatever be the sum indicated by a.

This section will serve to give the learner a general notion of the nature and use of Algebra, and the manner in which it is applied to the performing of questions.

SECTION II.

Algebraic Signs.

Besides letters, certain other signs are used in Algebra, some of which are also used in Arithmetic, though less frequently. It is by means of these and other arbitrary signs, that calculations in Algebra are per-

formed. They enable us to express a train of reasoning in a short and exact manner, so that the whole may be presented readily to the eye and the mind.

$+$ (*Plus*) *signifies addition;* as, $4 + 5$ is 9. This may be read, 4 *plus* 5 is 9, or 4 *and* 5 is 9, or in any other way which will indicate that 4 and 5 are to be added together.

Thus, too, the expression $x + y$ signifies that the quantities x and y are to be added, whatever their values may be. Suppose that the values of x and y are 6 and 4; then $x + y$ will be equal to $6 + 4$, or 10.

$-$ (*Minus,*) *placed before a quantity, signifies that it is to be subtracted.* Thus, $9 - 6$ is 3; which may be read, 9 *minus* 6 is 3, or 9 *less* 6 is 3, or 6 *from* 9 is 3, or in any other manner which will show that 6 is to be subtracted from 9.

So, too, in the algebraic expression $x - y$, the value of y is to be subtracted from the value of x. Suppose x, for instance, to be 12, and y to be 7; then $x - y$ will be the same as $12 - 7$, which is 5.

The signs $+$ *and* $-$ *affect only those numbers and quantities which immediately follow them.* Of course, the expressions,

$$9 + 6 + 5 - 7$$
$$9 + 6 - 7 + 5$$
$$9 - 7 + 6 + 5$$

are all of the same value, each being equal to 13; for the sign $+$ is *understood* before 9, the first number.

$=$ *Two horizontal lines signify equality;* that is, that the quantities between which they are placed, are equal to each other; as, $3 + 4 = 9 - 2$. If we

have $y + z = x$, we know that the value of x is equal to the values of y and z added together. Thus, f y be 6, and z be 4, the value of x must be 10; for $6 + 4 = 10$.

$\times$ *This sign signifies multiplication;* as, $3 \times 4 = 12$; that is, 3 multiplied by 4 is equal to 12. This character is often omitted when multiplication is implied; as in the expression $x\,y$, which is the product of x multiplied by y. Thus, if $x = 5$, and $y = 3$, $x\,y = 15$; for $3 \times 5 = 15$. But it is never omitted between two *numbers* which are to be multiplied.

$\div$ *This sign expresses that the quantity which precedes it is to be divided by that which follows it.* Thus, $12 \div 4 = 3$; that is, 12 divided by 4 is equal to 3.

But division is more frequently expressed in the form of a fraction; thus, $\frac{12}{4} = 3$, which may be read in the same manner.

So, too, $x \div y$, or $\frac{x}{y}$, expresses the quotient of x divided by y. Thus, if $x = 10$, and $y = 2$, $\frac{x}{y} = 5$; for $\frac{10}{2} = 5$.

A vinculum —— *is used to connect two or more quantities together.* Thus, $4 \times \overline{a + b}$ implies that the sum of a and b is to be multiplied by 4. Suppose the value of a to be 5, and of b to be 6; then,

$$4 \times \overline{5 + 6} = 44;$$

for $5 + 6 = 11$, and $4 \times 11 = 44$.

Again, $\overline{a + b} \times \overline{c + d}$ signifies that the sum of $a + b$ is to be multiplied by the sum of $c + d$. Let $a = 2$, $b = 3$, $c = 4$, and $d = 5$; then,

$$\overline{2 + 3} \times \overline{4 + 5} = 5 \times 9, \text{ or } 45.$$

A parenthesis ( ) *is often used instead of a vinculum, to indicate that several quantities are to be taken together.* Thus, $3\ (x+y)$ expresses that the sum of x and y is to be taken three times. If the value of x be 6, and of y be 4,

$$3\ (x+y) = 3\ (6+4) = 3 \times 10, \text{ or } 30.$$

The several quantities under a vinculum, or included in a parenthesis, may be taken collectively, and regarded as a simple quantity, of which the number prefixed is the coefficient.

SECTION III.

Simple, Compound, Similar, Positive and Negative Quantities.

A Simple quantity consists of a single term, that is, of one letter or number, or of several letters joined together without the sign + or —; as, x, $3\ y$, $a\,b\,c$, and $x\,y$, each of which is a simple quantity.

A Compound quantity consists of two or more simple quantities joined together by the sign + or —; as $x+y$, $a-b+3\ c$, $a\,b-7+z-8\,x$, each of which is a compound quantity.

A compound quantity, which consists of two terms only, as $x+y$, or $a-b$, is called a *binomial.* The latter expression, $a-b$, is also called a *residual* quantity, because it expresses the residue or remainder, after one of the terms has been taken from the other.

Similar quantities are such as differ only in their coefficients or signs. Thus, $3\ a$ and $5\ a$ are similar

quantities; so are $3\,x\,y$ and $7\,x\,y$; as also $2\,a\,b$ and $-8\,a\,b$; and the compound quantities $3\,a\,b + 4\,x$, and $5\,a\,b - 9\,x$.

All the quantities used in an algebraic calculation, are considered, in relation to each other, either as *positive* or *negative.*

A Positive quantity has the sign + prefixed to it; it is, in general, something to be added. When a positive quantity stands alone, as x, or is the first term of a compound quantity, as $a + b$, the sign is commonly omitted; but the sign + is always understood in such cases. Thus, x is the same as $+x$, and $a + b$ the same as $+a + b$.

A Negative quantity is one to be subtracted, and always has the sign — prefixed to it. Thus, in the expression $a - b$, $-b$ is a negative quantity, because its value is to be subtracted from a.

As the subject of positive and negative quantities is very apt to perplex beginners, a few examples will be given, by way of illustration.

1. William has 12 apples, and gives 5 of them to Samuel. How many has he left?

In this question, 12, the number of apples which William had in the first place, is a positive quantity; and 5, which must be subtracted to obtain the answer, is a negative quantity. $12 - 5$.

2. William has 12 apples, and Samuel gives him 5 more. How many has he then?

Here, as 5 must be added to 12 to obtain the answer, it is a positive quantity. $12 + 5$.

3. A man bought a watch for 25 dollars, and sold

it again for 30 dollars. How much did he gain by the bargain?

To find his gain, we must subtract what he gave for the watch from the sum for which he sold it: 30 is, therefore, a positive, and 25 a negative quantity $30 - 25$.

4. A man sold a watch for 30 dollars, by which bargain he gained 5 dollars. What did the watch cost him?

Here, the gain must be subtracted from the price of the watch: 5 is, therefore, a negative quantity. $30 - 5$.

5. A merchant went into trade with a certain sum, say a dollars; and, at the end of the year, he found himself worth b dollars. How much did he gain during the year?

We must subtract what he had at the beginning of the year, a dollars, from what he had at the end of it, b dollars, to ascertain his gain: a is, therefore, a negative quantity, and the state of his affairs may be expressed thus, $b - a$.

In this question, if we suppose the merchant to have *lost* instead of *gained* by his business, it is evident that the value of b will be less than that of a, and we shall be required to subtract a greater number from a less, which is impossible. But it is perfectly easy to represent such a subtraction, as, for instance, $18 - 32$; and hence it frequently happens in Algebra, that a negative quantity stands alone, as $-x$, when there is no quantity from which it is to be actually taken.

6. A man has in his possession 200 dollars, and owes debts to the amount of 500 dollars. How much is he worth?

Here, the money the man has is a positive quantity, and the amount of his debts, which is to be subtracted, is a negative quantity; therefore, the expression 200 — 500 will represent the state of his property. Now, if he pay off his debts, as far as his 200 dollars will go, there will still be $ — 300 left; that is, he will be 300 dollars worse than nothing.

In the last question, the amount of the *debts* might be regarded as the positive quantity; and then the opposite quantity, the money on hand, would be negative, and 500 — 200 would represent the amount of debts which the man could not pay.

It is evident, therefore, that *positive* and *negative* are merely relative terms, which are, in general, opposed to each other. In any calculation, whatever quantity is assumed as positive, all other quantities of a similar nature, or which tend to *increase* it, are also positive; and whatever quantities are opposed to it, in any way, or which serve to *diminish* it, are negative.

CHAPTER II

ADDITION.

SECTION I.

Simple Quantities that are Similar.

1. A man gave to one poor person a dollars, to a second $3\,a$ dollars, and to a third $4\,a$ dollars. How much did he give them all? ANS. $8\,a$ dollars.

In this question, we have three *simple* quantities: they are all *similar*, and they are all *positive*; for it must be remembered that, when no sign is used, the sign $+$ is always understood. The first quantity is a; the second, $3\,a$, which may be written $a + a + a$; and the third, $4\,a$, or $a + a + a + a$. Now, by counting, we find there are eight a's or $8\,a$. But the sum of the coefficients, $1 + 3 + 4$, is also 8. Then, to perform questions of this kind, *Add together all the coefficients, and place the sum before the common letters.*

Suppose the value of a, in this question, to be 5; then we shall have

$$\begin{array}{r} a = 1 \times 5 = 5 \\ 3\,a = 3 \times 5 = 15 \\ \underline{4\,a} = 4 \times 5 = \underline{20} \\ 8\,a = 8 \times 5 = 40 \end{array}$$

By assigning any other value to the letter a, the student will obtain a similar result.

2.) $4\,b$	3.) $x\,y$	4.) $5\,a\,b\,c$	5.) $7\,y\,z$
$3\,b$	$2\,x\,y$	$6\,a\,b\,c$	$y\,z$
$5\,b$	$7\,x\,y$	$a\,b\,c$	$6\,y\,z$
$6\,b$	$3\,x\,y$	$2\,a\,b\,c$	$2\,y\,z$
$18\,b$	$13\,x\,y$		

6. Add together $a\,x$, and $3\,a\,x$, and $5\,x\,a$.

It is of no consequence in what *order* the letters are given; for $5\,x\,a$ is evidently of the same value as $5\,a\,x$. Let $x = 6$, and $a = 4$, for instance; then $5\,a\,x$ will be $5 \times 4 \times 6$, or 120; and $5\,x\,a$ will be $5 \times 6 \times 4$, which is also equal to 120. It is usual, however, to *arrange the letters according to their order in the alphabet.*

7. Add together $3\,a\,b\,c$, and $2\,a\,b\,c$, and $4\,a\,b\,c$.

8. What is the sum of $5\,a\,r\,x$, and $3\,a\,r\,x$, and $a\,r\,x$; and $7\,a\,r\,x$, and $17\,a\,r\,x$?

9. A merchant is indebted $4\,a$ dollars to A, $5\,a$ dollars to B, $6\,a$ dollars to C, and $8\,a$ dollars to D. How much does he owe them all?

10. What is the amount of $a\,b\,c$, and $5\,a\,b\,c$, and $7\,a\,b\,c$, and $12\,a\,b\,c$?

The student will find it a useful exercise to *prove* his answers, by assigning definite values to the letters given in the questions, in the manner exhibited above.

11. Add together $-x$, and $-3\,x$, and $-5\,x$.

Ans. $-9\,x$.

This example differs from the foregoing in only one particular; that is, the quantities are all *negative:*

the sign — must, therefore, be prefixed to the sum; for the *whole* must evidently be of the same character as the *parts* of which it is composed.

Let us suppose that x, in this example, stands for 100 dollars; then $-x = -100$, $-3x = -300$, $-5x = -500$, and the answer $-9x = -900$ dollars. But it may be asked, How can 100, or 300, or 500, or 900 dollars be subtracted from nothing? Such a subtraction *may be represented*, although it cannot be *performed*.

A merchant, for instance, wishes to ascertain the profits of his business. His *gains* are positive quantities; and his *losses*, because they must be subtracted from his gains before his clear profits can be known, must be negative. Now, it is evident, if he has lost 100 dollars by one speculation, 300 dollars by another, and 500 dollars by bad debts, that these sums should be written -100, -300, and -500; and that they may be added together, as if they were positive quantities, their amount being written -900, to show that it is to be subtracted from some other quantity; that is, from the amount or sum total of his profits.

12. Add together $-6xy$, $-xy$, and $-16xy$.

13. What is the amount of $-7abc$, $-12abc$, $-abc$, $-6abc$, and $-24abc$?

14.)	15.)	16.)
$-7xyz$	$-12abx$	$-7abcd$
$-3xyz$	$-4abx$	$-12abcd$
$-4xyz$	$-6abx$	$-16abcd$
$-13xyz$	abx	$-5abcd$

17. What is the sum of $-5\,a\,b\,x$, and $-3\,a\,b\,x$, and $-7\,a\,b\,x$, and $-2\,a\,b\,x$?

18. What is the sum of $-12\,a\,b\,x\,y$, $-7\,a\,b\,x\,y$, $-17\,a\,b\,x\,y$, $-a\,b\,x\,y$, and $-2\,a\,b\,x\,y$?

19. Add together $-3\,a$, and $-7\,a$, and $-12\,a$, and $-16\,a$.

20. What is the sum of $-x\,y$, and $-3\,x\,y$, and $-7\,x\,y$, and $-12\,x\,y$?

21. Add together $3\,x$, and $-2\,x$. ANS. x.

This example contains both a *positive* and a *negative* quantity. If the Dr. column of a leger amount to $3\,x$ dollars, and the Cr. column to $2\,x$ dollars, the condition of the account may be expressed $3\,x - 2\,x$; and the sum due is evidently x dollars.

Suppose the value of x to be 10 dollars; then $3\,x = 30$, and $-2\,x = -20$, and the account will be Dr. 30 dollars, Cr. 20 dollars, due 10 dollars. The 20 dollars credited will *cancel* the same amount of the debt; that is, $-2\,x$ will cancel $+2\,x$, leaving $+x$ due.

22. Add together $2\,x$, and $-3\,x$. ANS. $-x$.

This example is like the last, only the *negative* is the *larger* quantity, and the answer must have the sign $-$ prefixed. For it is evident, if the Dr. side of an account be $2\,x$, or 20 dollars, and the Cr. side be $3\,x$, or 30 dollars, the debt has all been paid, and x, or 10 dollars more, which is to be paid back.

From these examples we derive the following RULE for adding together two similar quantities, when their signs are not alike: *Subtract the less coefficient from*

the greater. The answer must have the same sign as the greater quantity.

Hence it appears, that what is called Addition in Algebra, is sometimes performed by Subtraction. This is an apparent contradiction, which is very apt to perplex beginners. The whole difficulty will vanish, however, if they remember the object of Addition; which is, to express the *value* of two or more quantities, of any kind, in the most simple manner possible.

23. What is the sum of $5\,x\,y$, and $-3\,x\,y$?

Ans. $2\,x\,y$.

24.) $\begin{array}{r} +4\,a\,b \\ -3\,a\,b \\ \hline \end{array}$ 25.) $\begin{array}{r} +7\,a\,b\,x \\ -5\,a\,b\,x \\ \hline \end{array}$ 26.) $\begin{array}{r} +16\,a\,y \\ -16\,a\,y \\ \hline \end{array}$

27. Add together $19\,a\,b\,x\,m$, and $-27\,a\,b\,m\,x$.

28. What is the sum of $-3\,a\,m\,x$, and $5\,a\,m\,x$?

29. What is the sum of $5\,x$, and $-3\,x$, and $4\,x$, and $-2\,x$? Ans. $4\,x$.

Let us suppose the *positive* quantities in this question, namely, $5\,x$ and $4\,x$, to represent sums of money received, and the *negative* quantities, $-3\,x$ and $-2x$, payments made. Then, $9\,x$ dollars are received, and $5\,x$ dollars paid away; and, of course, $4\,x$ dollars remain on hand.

If the value of x be 100 dollars, then 900 dollars are received, 500 dollars paid away, and a balance of 400 dollars remains.

30. Add together $4\,x$, and $-8\,x$, and $-7\,x$, and $6\,x$. Ans. $-5\,x$.

This example is like the last; but here, the sum of the negative quantities is greater than that of the posi

tive, and the difference is marked with the sign —. The sum of $4x$ and $6x$ is $10x$; $-8x$ and $-7x$ is $-15x$; the difference, *or balance between the two sums*, is $-5x$.

If a trader gain 40 dollars in one week, lose 80 dollars the next, lose 70 dollars the third, and gain 60 dollars the fourth, in the four weeks he gains 100 and loses 150 dollars; that is, he loses 50 dollars more than he gains, and his amount of loss and gain stands — 50. In this case, we suppose the value of x to be 10 dollars.

31. Add together $5abc$, $-7abc$, $3abc$, $-abc$, and $2abc$. Ans. $2abc$.

The sum of the positive quantities is $10abc$; and that of the negative quantities is $-8abc$; and $10abc - 8abc = 2abc$.

When several similar quantities, some of which are + and others —, are to be added, it will generally be found the most convenient way to proceed as in the last three examples, namely: *Reduce them to two terms, by bringing all the positive quantities into one sum, and all the negative quantities into another, and then balance these two terms in the usual manner.*

32. Add together $8x$, and $-3x$, and $4x$, and $-6x$.

33.) $-7ab$	34.) $-2bc$	35.) $abcd$
$3ab$	$5bc$	$-6abcd$
$6ab$	$-4bc$	$-8abcd$
$-ab$	$6bc$	$4abcd$

36.)	37.)	38.)
$x\,y$	$73\,b\,c\,x$	$-\;m\,x\,y$
$3\,x\,y$	$-60\,b\,c\,x$	$16\,m\,x\,y$
$7\,x\,y$	$2\,b\,c\,x$	$-20\,m\,x\,y$
$10\,x\,y$	$-15\,b\,c\,x$	$2\,m\,x\,y$

39. Add together $4\,a\,x$, and $5\,a\,x$, and $-3\,a\,x$, and $7\,a\,x$, and $-6\,a\,x$, and $-2\,a\,x$, and $9\,a\,x$, and $-19\,a\,x$.

40. Add together $14\,a\,b\,x$, and $-6\,a\,b\,x$, and $-7\,a\,b\,x$, and $3\,a\,b\,x$, and $7\,a\,b\,x$, and $-5\,a\,b\,x$, and $3\,a\,b\,x$, and $-2\,a\,b\,x$, and $-7\,a\,b\,x$, and $3\,a\,b\,x$, and $7\,a\,b\,x$, and $-a\,b\,x$, and $-8\,a\,b\,x$.

SECTION II.

Simple Quantities that are Dissimilar.

1. What is the amount of $4\,a$, and $3\,x$, and $5\,y$?

ANS. $4\,a + 3\,x + 5\,y$.

In this question, the quantities are *dissimilar*, that is, different from each other; and, of course, they cannot be actually added. All that can be done is, to *connect them together by their proper signs.*

2. What is the sum of $3\,b$, and $7\,c$, and $-5\,x$?

ANS. $3\,b + 7\,c - 5\,x$.

3. Add together $6\,x$, and $-5\,y$, and $7\,x\,y$, and $-4\,z$.

4. Add together $7\,a\,b$, and $-4\,y$, and $12\,b\,x$, and -6.

5.)	6.)	7.)	8.)
$6abx$	$19ax$	$-4y$	ar
$-5a$	$-3amn$	z	$-3z$
$-xy$	xyz	axy	$-at$
27	$-am$	$-mn$	-61

9. What is the sum of $4ab$, and $-5ax$, and $-3ab$, and $6ax$, and abc, and $-ab$, and -16

Ans. $ax + abc - 16$.

In this question, we have $+4ab$; also $-3ab$ and $-ab$, which, together, make $-4ab$. As these two quantities, $+4ab$ and $-4ab$, *cancel* each other, they do not appear in the answer. We also have $-5ax$ and $+6ax$, which balance each other in part, the difference between them being $+ax$, which is the first term of the answer. The remaining terms, $+abc$ and -16, being dissimilar, are annexed.

In all cases, the ADDITION *is performed, if we connect together the several given quantities by their respective signs.* Thus, $4ab - 5ax - 3ab + 6ax + abc - ab - 16$, is a true answer to the last question, and expresses the value of the several quantities as perfectly as the answer given above. What remains is a species of *reduction,* by which we unite together as many of the terms as we can, for the sake of diminishing their number, and rendering the answer more simple.

It is evident that *two equal similar quantities whose signs are unlike, as $+5x$ and $-5x$, cancel each other.*

10 What is the sum of $16ab$, and $-3ax$, and

$a\,b\,c$, and $-12\,a\,b$, and $3\,a\,x$, and -75, and 40 and $-4\,a\,b$? ANS. $a\,b\,c - 35$.

11.) $\begin{array}{r} -4\,a\,b\,c \\ 6\,a\,x\,y \\ 7\,a\,b\,c \\ -\;x\,y \\ \hline \end{array}$ 12.) $\begin{array}{r} 12\,a\,x \\ -\;a\,y \\ 3\,a\,x \\ -14\,a\,x \\ \hline \end{array}$ 13.) $\begin{array}{r} -5\,x \\ -2\,z \\ 3\,x \\ a\,b\,c \\ \hline \end{array}$

14. Add together $5\,a\,b\,c$, and $12\,x$, and $-14\,a\,x$, and $-4\,a\,b\,c$, and $-16\,x$, and $3\,y$, and $-5\,a\,x$, and 76, and $-a\,b\,c$, and $5\,x$, and 83, and $16\,a\,x$.

15. What is the sum of $15\,a\,x\,y$, and $a\,b$, and $-2\,b\,m\,n$, and $-12\,a\,x\,y$, and -24, and $3\,b\,m\,n$, and 17, and $-a\,b$, and $3\,a\,x\,y$, and $y\,z$?

16. What is the sum of $14\,a\,x$, and $3\,b$, and $2\,x$, and $17\,y$, and 37, and $-4\,b$, and $-15\,a\,x$, and $-12\,y$, and $a\,x$; and 37, and $-5\,y$, and -74, and $-x$?

17. Add together $6\,a\,b$, $5\,x\,y\,z$, 14, $-8\,a\,b$, 21, $-3\,x\,y\,z$, $c\,d$, $4\,a\,b$, -35, and $-2\,x\,y\,z$.

18. What is the amount of $8\,x\,y$, $-5\,b\,c\,d$, 18, $-3\,x\,y$, -9, $b\,c\,d$, -12, and $4\,b\,c\,d$?

19.) $\begin{array}{r} -3\,h\,m \\ 54 \\ +9\,h\,m \\ -60 \\ \hline \end{array}$ 20.) $\begin{array}{r} -7\,x\,y\,z \\ -3\,m\,n \\ 8\,y\,z \\ 2\,m\,n \\ \hline \end{array}$ 21.) $\begin{array}{r} 754 \\ a\,b\,x \\ -649 \\ m \\ \hline \end{array}$

22.) $\begin{array}{r} 8\,x\,y\,z \\ -\;x\,y \\ -3\,x\,y\,z \\ \hline \end{array}$ 23.) $\begin{array}{r} -\;a\,b\,x \\ -723 \\ 5\,a\,b\,x \\ \hline \end{array}$ 24.) $\begin{array}{r} 9\,y\,z \\ -7\,y\,z \\ y\,z \\ \hline \end{array}$

SECTION III.

Compound Quantities.

1. Add together $3a + 2b$ and $5x + 3y$.

Ans. $3a + 2b + 5x + 3y$.

In this example, *compound* quantities are given; but they are added in the same manner as simple quantities; that is, *all the terms of which they are composed are connected together with their proper signs.* If we were required to add together $4 + 5$ and $3 + 7$, we might either say, $4 + 5 + 3 + 7 = 19$; or $4 + 5 = 9$, and $3 + 7 = 10$, and then add these two sums, $9 + 10 = 19$. The result, both ways, is the same; but, in Algebra, the latter mode is not practicable, unless the quantities are similar.

2. Add together $5b + 2c$, and $4d - 3y$.

Ans. $5b + 2c + 4d - 3y$.

This example is like the last, excepting that one of the terms is affected with the sign —, which must be retained in the answer; for we are not required to add the whole value of $4d$ to $5b + 2c$, but only the difference between $4d$ and $3y$; and when we add $4d$, as in the answer above, we add $3y$ too much, which must be subtracted.

This may be rendered more intelligible, perhaps, by numbers. Add together $3 + 4$ and $8 - 6$. First, $3 + 4 = 7$, and $8 - 6 = 2$; and $7 + 2 = 9$, which is the amount of the numbers given. Again, $3 + 4 + 8 = 15$, which is too much by 6, which must be subtracted: thus, $3 + 4 + 8 - 6 = 9$, as before.

3. Add together $5a + 2c$, and $3x - 4y$, and $2b - z$. Ans. $5a + 2c + 3x - 4y + 2b - z$.

4. Add together $5a + 2b + d$, and $3x - 2b$, and $4d - 3x - 6a$. Ans. $5d - a$.

By adding all the terms in these several quantities, we obtain

$$5a + 2b + d + 3x - 2b + 4d - 3x - 6a.$$

By cancelling $+ 2b$ and $- 2b$, and $+ 3x$ and $- 3x$,

$$5a + d + 4d - 6a.$$

By adding the d and $4d$,

$$5a + 5d - 6a.$$

By balancing the $+ 5a$ and $- 6a$,

$5d - a$, as above.

The answer should always be reduced, in this manner, to the least number of terms possible.

5. What is the sum of $5ax + 3bc$, and $7ax - 4bc$, and $- 3ax + 17y$?

6. Add together $3(a - b)$, and $2(a - b)$.

Ans. $5(a - b)$.

In this question, $3(a - b) = 3a - 3b$; and $2(a - b) = 2a - 2b$; and the answer, $5(a - b) = 5a - 5b$. If $a = 4$, and $b = 2$, $(3a - 3b) = (12 - 6) = 6$; and $(2a - 2b) = (8 - 4) = 4$; and $6 + 4 = 10$: but $(5a - 5b) = (20 - 10) = 10$ also. *When the compound quantities, included in parentheses, are alike, they are added like simple quantities that are similar.* The numbers or letters before the parentheses, are regarded as coefficients, and added as such; but the quantities included undergo no change.

7. What is the sum of $6\,(5\,a - x\,y)$, and $3\,(5\,a - x\,y)$?

8. Add $4\,(a\,b - x + 3\,y)$, $2\,(a\,b - x + 3\,y)$, and $5\,(a\,b - x + 3\,y)$, together.

9. Required the sum of $3\,a\,(m\,n - 6 + y\,z)$, $2\,a\,(m\,n - 6 + y\,z)$, and $4\,a\,(m\,n - 6 + y\,z)$.

10. Add together $2\,(b\,c - x\,y)$, $3\,(b\,c - x\,y)$, $4\,(b\,c - x\,y)$, $5\,(b\,c - x\,y)$ and $2\,(b\,c - x\,y.)$

From the several examples which have been given in the course of this Chapter, and the explanations with which they have been accompanied, may be derived the following general RULE for performing all questions in Addition. *Connect together all the terms of the given quantities, by their proper signs, and unite such as are similar.*

11. Add together $3\,a\,b + 2\,c$, and $5\,a\,x - c + 16$, and $14 - 3\,a\,x$, and $5\,a\,b + 4\,c - y$.

12. Add together $9\,x - 4\,y + 6\,z - m\,n + 8 - 2\,b + 7 + 4\,b - 3\,x + 7\,y - 4\,m\,n - x - 8\,b + 7 - 5\,z + 2\,y + 6\,x + z + 9\,b - 18 + 7\,m\,n - 10\,x - 4 - 3\,b - 5\,y - 2\,z - m\,n$.

This example may be conveniently arranged for adding, in the following manner:—

$$\begin{array}{rrrrrr}
9\,x & -\,4\,y & +\,6\,z & -\,m\,n & +\,8 & -\,2\,b \\
-\,3\,x & +\,7\,y & & -\,4\,m\,n & +\,7 & +\,4\,b \\
-\,x & +\,2\,y & -\,5\,z & & +\,7 & -\,8\,b \\
6\,x & & +\,z & +\,7\,m\,n & -\,18 & +\,9\,b \\
-\,10\,x & -\,5\,y & -\,2\,z & -\,m\,n & -\,4 & -\,3\,b \\
\hline
\end{array}$$

13. Add together $5bc + 2ax - 3y - 12 + 6acx - 8ax + 2y - 6bc + 18 - 9acx + 14 + 9bc - 8ax - 7y + 17 + 7bc - 12y - 18 + 7acx - 6y + ax$.

14.)
$$\begin{array}{l} 8x - 7y + 6m + 4 - 2x \\ -4y - 9 - 2x - 6z + 9m \\ 7m - 9x + 12 - 8y + 4z \\ 10x - 4m + 3y - 6 + 4x \\ \hline \end{array}$$

15.)
$$\begin{array}{l} 8abc - 5a(b-x) - 5 + xyz \\ 3a(b-x) - 4abc + 4xyz - 10 \\ 2abc + 9 - 15xyz + 24 \\ 7xyz - 2abc + 4a(b-x) - 18 \\ \hline \end{array}$$

16.)
$$\begin{array}{l} 5a - 17 - 7a + 3xy \\ 4(x-y) - 13 - 10xy + 21 \\ 8a + 8(x-y) - 2xy + a(x-y) \\ 8xy + 9 - 3a - 10(x-y) \\ \hline \end{array}$$

17.)
$$\begin{array}{l} 16ab - 3xz + 9 - 5m \\ 9xz - 3 + m - 9ab \\ 4 - 3m - 18ab - 8xz \\ 6m + 12ab + 2xz - 11 \\ \hline \end{array}$$

18.)
$$\begin{array}{l} 9axy - 15 + a(a-b) - 9bc \\ bc - 8a(a-b) + 36 - 5axy \\ 4a(a-b) - 5bc + axy - 13 \\ 2 - 4axy + 8bc + 5a(a-b) \\ \hline \end{array}$$

CHAPTER III

SUBTRACTION.

SECTION I.

To subtract a Simple Quantity.

1. If a man receive $5\,x$ dollars, and pay away $3\,x$ dollars, how much has he left? Ans. $2\,x$ dollars.

In this example, *where the quantities are similar, we obtain the answer by subtracting the coefficients.* The work is written thus, $5\,x - 3\,x$, and then the quantities are balanced; that is, the terms are reduced, according to the directions given in Addition. Let it be observed, that *the sign of the quantity to be subtracted, is changed from + to —.* But the sign of the quantity from which the subtraction is made, is not changed.

Suppose the value of x to be 5; then $5\,x = 25$, and $3\,x = 15$; and $25 - 15 = 10$, or $2\,x$, as above.

2.) From $3\,x$	3.) $5\,b$	4.) $16\,y$	5.) $9\,x$
Take x.	$2\,b$.	$4\,y$.	$2\,x$.
$2\,x$			

6.) From $16\,a\,b\,c$	7.) $4\,x\,y\,z$	8.) $14\,m\,y$
Take $2\,a\,b\,c$.	$x\,y\,z$.	$3\,m\,y$.

9. Subtract $5\,a\,x$ from $7\,a\,x$.

10. Take $9\,x\,y\,z$ from $10\,x\,y\,z$

11. Take $6\,a\,b\,x$ from $6\,a\,b\,x$.
12. From $19\,h\,m\,p$ take $7\,h\,m\,p$.

13. Subtract b from a. ANS. $a - b$.

The quantities a and b being dissimilar, we are obliged to *represent* the subtraction, as in the answer given.

That the above expression, $a - b$, contains the true answer, may be proved by referring to numbers. Let $a = 12$, and $b = 8$; then $a - b$, that is, $12 - 8 = 4$, which is the difference of the given quantities. Here, too, *the sign of the number to be subtracted is changed from + to —.*

14. From $5\,x$ take $2\,y$. ANS. $5\,x - 2\,y$.

15.)	16.)	17.)	18.)
From $5\,a$	$7\,x$	$2\,m$	$7\,x$
Take b.	$3\,y$.	$3\,n$.	$2\,y$.
$5\,a - b$.			

19.)	20.)	21.)
From $14\,a\,b\,c$	$12\,a\,x\,z$	$19\,a$
Take $2\,x\,y$.	$29\,h\,m$.	$2\,y$.

22. Subtract $5\,x\,y$ from $4\,z$.
23. Take $13\,b\,c$ from x.
24. From $7\,m\,x$ take p.
25. Subtract $m\,n$ from $b\,c$.
26. From x take $13\,b\,c$.
27. Take $7\,m\,x$ from p.
28. Subtract $b\,c$ from $m\,n$.
29. From $x\,y\,z$ take $x\,y$.
30. Take 16 from $32\,a\,b\,c$.
31. From $8\,a\,b\,c\,x$ take $4\,a\,b\,c$.

SECTION II.

To subtract a Compound Quantity.

1. A man, who has $4\ x$ dollars in his pocket, pays one debt of $3\ x$ dollars, and another debt of y dollars How many has he left? Ans. $x - y$ dollars.

It is here required to take the *whole* value of the compound quantity, $3\ x + y$, the sums paid away, from $4\ x$. Now, if only $3\ x$ be taken, it is evident that not enough is subtracted by the value of y, whatever that may be. The work may, therefore, be expressed thus, $4\ x - 3\ x - y$; which, reduced, gives $x - y$.

To illustrate this by figures, let $x = 5$, and $y = 3$: then $4\ x = 20$, and $3\ x + y = 15 + 3$, or 18: now, $20 - 18 = 2$, and $20 - 15 - 3$ is also 2. In this last expression, we may either subtract 15 from 20, and then subtract 3 from the remainder; or we may add -15 and -3 together, and subtract their sum from 20. The result is the same both ways. Here, *the signs of both the quantities to be subtracted are changed from + to —.*

2.) Subtract $3\ a + b$ from $5\ x$. Ans. $5\ x - 3\ a - b$

3.) From $3\ a$
Take $2\ a + b$.
$a - b$.

4.) $5\ a\ b\ c$
$a\ b\ c + y$.

5.) $4\ x$
$b + z$.

6.) From $a\ b\ c\ d$
Take $2\ m + 3\ y$.

7.) 47
$x + b\ d$.

8.) $a\ b\ x$
$3\ a\ x + 12$.

9. From $6a$ take $3a + b$.
10. Subtract $2x + y$ from $4a$
11. Take $x + 16$ from x.
12. Take $23 + x$ from 40.
13. Subtract $4y + x$ from bc.
14. From y take $y + x$.
15. Take $4 + x$ from 5.
16. From $5xy$ take $xy + 5$.
17. Subtract $36 + 2y$ from 48.
18. Take $x + y$ from $2x$.

19. From $5x$ take $3x - y$. ANS. $2x + y$.

In this example, the value of the compound quantity $3x - y$ is to be taken from $5x$. The *whole* value of $3x$ is not to be subtracted, but the *difference* between that value and the value of y. If, therefore, we subtract the whole of $3x$, we subtract too much by the value of y, which must afterwards be added, to give the true answer. The work may be expressed thus, $5x - 3x + y$; which, reduced, is $2x + y$.

Perhaps this will be better understood, if illustrated by figures. Let $x = 6$, and $y = 4$; then $5x = 30$, and $3x - y = 18 - 4$; that is, we are required to take $18 - 4$ from 30. Now, $18 - 4 = 14$, and $30 - 14 = 16$, which is the true answer. But if we take the whole of 18 from 30, we take too much by 4, as we are required to subtract only the *excess* of 18 over 4; we must, therefore, add 4 to the remainder, to obtain the true answer; thus, $30 - 18 + 4 = 16$. We may either add 4 to 30, and subtract 18 from the sum; or we may subtract 18 from 30, and add 4 to

the remainder. Here, too, *both the signs of the quantities to be subtracted are changed, the* $+$ *to* $-$ *and the* $-$ *to* $+$.

20.) From $14\,ab$
Take $\underline{12\,ab - xy - z.}$
$2\,ab + xy + z.$

21.) $17\,xy$
$\underline{12\,xy - abm.}$

22.) From $5\,ab$
Take $\underline{5\,ab - 7 + x.}$

23.) $27\,mx$
$\underline{19\,mx - y - 3\,ab.}$

24. Subtract $bc - 8$ from 16.
25. Take $xyz - a$ from bcx.
26. From ab take $y - x$.
27. From 48 take $x - 16$.
28. Take $xz - y$ from bx.
29. Subtract $xyz - 9\,x$ from abc.
30. From y take $y - 4$.
31. Subtract $ab + 7$ from ax.
32. Take $xy - 9\,a$ from $2\,xy$.
33. From $3\,ad$ take $2\,ad + 9$.
34. Subtract $5\,m - 5\,a$ from $4\,m$.
35. From 18 take $x - 18$.
36. Take $a + b$ from a.
37. Subtract $12 + 4\,a$ from 27.
38. Subtract $a + 12$ from 19.
39. From $5\,(a + b)$ take $2\,(a + b) - x$.
40. From $x\,(x - y)$ take $2\,x\,(x - y) - xy$
41. Take $ax + b$ from $3\,ax$.

SECTION III.

General Rule for Subtraction.

1. From x subtract $-x$. Ans. $2x$.

According to the principles already explained, $-x$ becomes $+x$, when it is subtracted from any quantity; we have, therefore, $x + x = 2x$; that is, *subtracting a negative quantity is the same thing as adding a positive quantity of the same value.* If A is in debt 1000 dollars, we should subtract that sum in forming an estimate of his property; but if B cancels that debt for him, that is, subtracts that — quantity, he evidently increases or adds to the amount of his property as much as if he had actually given 1000 dollars into his hand.

2. From $a + b$ subtract $x - y$.

Ans. $a + b - x + y$.

It is here required to subtract the *difference* of two quantities, x and y, from the *sum* of two other quantities, a and b. Suppose $a = 8$, $b = 6$, $x = 11$, and $y = 2$: we then have $8 + 6$, from which we are to subtract $11 - 2$; that is, $14 - 9$, or 5, which is the answer. But $8 + 6 - 11 + 2$ is also equal to 5, which corresponds with the answer as expressed above. The signs of the quantities subtracted are changed as before; but in all cases, *the signs of the other quantities, from which the subtraction is made, remain unchanged.*

From the several questions proposed in this chapter,

and the reasoning which follows them, we derive the following general RULE for Subtraction in Algebra: *Change all the signs in the quantity to be subtracted, each + to — and each — to +; and unite the terms that are similar, as in Addition.*

The subtraction is, in fact, performed, when the signs of the terms to be subtracted are changed. The object of the remaining part of the operation, is, to reduce the number of terms, by uniting or cancelling such as are similar, that the answer may be presented in its simplest form.

3. What is the value of $x - (a + b - c)$?

ANS. $x - a - b + c$.

The expression used in this question implies, that the whole quantity included in the parenthetical marks ( ), namely, $a + b - c$, is to be subtracted from x; of course, all the signs must be changed.

4. What is the value of $ab - (-cx + d - 16)$?

ANS. $ab + cx - d + 16$.

5.) From $6x + 3y$
Take $5x - 4y$

6.) $17xyz + 14 - ab$
$13ab + 24 + 18xyz$.

It is recommended to the student, in performing these examples, actually to change the signs; at least, until he becomes perfectly familiar with the operation. The last two examples, thus prepared, will stand

5.)
$$\begin{array}{r} 6x + 3y \\ -5x + 4y \\ \hline x + 7y \end{array}$$

6.)
$$\begin{array}{r} 17xyz + 14 - ab \\ -13ab - 24 - 18xyz \\ \hline -14ab - 10 - xyz. \end{array}$$

7.) From $17xy - 102 - 12abm$
Take $47abm - xy + 3ar$.

8.) $7xm + 14y$
$14y + 6m$.

9.) From $abx + 3yz$
Take $3yz + abx$.

10.) $21xyz - 34 + z$
$21 + z - 14xyz$.

11.) From $an + op$
Take $xyz - 60$.

12.) $19abx - 2amn - 7bcd$
$3abc + 5bcd - amn$.

13.) From $x + y$
Take $x - y$.

14.) $ryz - 15 + 7ab + 25x$
$24 + 2ryz - 8x$.

15.) From $24x + 3yz - 12abc - 6mnp + 5$
Take $16abc - 6mnp + 23x + yz + 12 - xy$.

16.) From $x + y + z$
Take $x - y - z$.

17.) $3abx - 14 + 6a$
$3abx - 14 - 6a$.

18.) From $3x + 5y - z$
Take $2x + 5y + z$.

19.) $8am - xy$
$3am - z$.

20.) From $6mp - 8az + 12 - 14mxy$
Take $-6mp + 8az - 12 + 14mxy$.

21.) From $3ax + 5ay - 2ab$ take $2ax + 5ay - 3ab$.

22.) Take $18 + 2am$ from $5am - 17$.

CHAPTER IV.

MULTIPLICATION.

SECTION I.

Simple Quantities.

1. What will 5 oranges come to, at a cents apiece?

Ans. 5 a cents.

It is evident that 5 oranges must cost five times as much as one orange; therefore, if one orange cost a cents, whatever the value of a may be, 5 oranges will cost 5 a cents. Let $a = 2$ cents; then $5\,a = 5 \times 2$ or 10 cents.

MULTIPLICATION *is merely a short way of performing* ADDITION, *when the quantities to be added happen to be equal.* When these quantities are unequal, their united value or amount can be found only by adding them all together. And it is evident that, when they are equal, their amount *can* be found in the same way. Thus,

$$
\begin{array}{lll}
2+2=4, & \text{or} & 2\times 2=\ \ 4, \\
2+2+2=6, & \text{or} & 2\times 3=\ \ 6, \\
2+2+2+2=8, & \text{or} & 2\times 4=\ \ 8, \\
2+2+2+2+2=10, & \text{or} & 2\times 5=10;
\end{array}
$$

where, in the *first* column, the same results are ob

tained by addition, as are obtained, in the *second*, by multiplication.

So, too, in *literal* quantities; if one orange cost a cents, two oranges will cost $a + a = 2\,a$; three, $a + a + a = 3\,a$; four, $a + a + a + a = 4\,a$; and five, $a + a + a + a + a = 5\,a$ cents, as in the question.

To multiply a literal quantity by a number, therefore, all that is required is, *to make the number the coefficient of the quantity*; for the coefficient shows how many times the value of the quantity is to be taken.

In Algebra, as in Arithmetic, the number to be multiplied is called the *multiplicand*; the number by which we multiply, the *multiplier*; and the result of the operation, the *product*. The multiplier and multiplicand, when spoken of together, are called *factors*.

2. If a yard of cloth is worth a dollars, what is the value of 6 yards? ANS. \$6 a.

It is recommended to the student to *prove* his answers to the questions, by substituting such numbers for the letters given, as he pleases.

3. If there be x apples in a bushel, how many are there in 9 bushels? ANS. 9 x apples.

4. If the interest of a given sum of money be y dollars per annum, what will the interest be for 8 years?

5. How many bushels of corn are there in a field of 12 acres, which produces a bushels to the acre?

6. How much will 10 yards of cloth come to, at r dollars per yard?

7. A man gave away a dollars every day; how much did he give away in a week?

8. What are a man's expenses for a year, who expends a dollars a day?

9. If there be x yards of cloth in one piece of linen, how many yards are there in 15 pieces?

10. How many rods of wall can a man build in three weeks, if he build a rods a day?

11. How many panes of glass are required for 14 windows, each window having c panes?

12. What is the value of a yards of calico, at c cents a yard? ANS. $a\,c$ cents.

It is evident that the price, c, must be multiplied by the quantity, a; which may be expressed thus, $a \times c$, or thus, without the sign, $a\,c$. Let $a = 7$, and $c = 12$; then $a \times c$, or $a\,c = 7 \times 12$, or 84.

If the price had been $c\,d$ cents, we should have multiplied that quantity by a, and the answer would have been $a\,c\,d$.

Or, if the quantity had been $a\,b$ yards, and the price $c\,d$ cents, we should still have multiplied the one by the other, and the answer would have been $a\,b\,c\,d$.

Hence, to multiply one simple literal quantity by another, we *write all the letters in both quantities together*.

13. If x men can do a piece of work in y days, how long will it take one man to do it? ANS. $x\,y$ days.

14. Multiply $a\,x$ by y. ANS. $a\,x\,y$.

15. Multiply $b\,c$ by $m\,n$.

16. Multiply $a\,b\,x$ by $y\,z$.

17. Multiply $a\,m\,n$ by $x\,y\,z$.

18. Multiply $a\,b\,c$ by $h\,m\,n$.
19. Multiply x by $a\,b\,c$.
20. Multiply $m\,x\,z$ by $a\,b\,y$.

21. If a horse travel b miles in one hour, how far will he travel in $2\,a$ hours? ANS. $2\,a\,b$ miles.

Let $b = 6$, and $a = 5$; then $2\,a = 2 \times 5$, or 10; and $2\,a\,b = 2 \times 5 \times 6$, or 60. Or we may say, in a hours he will travel $a\,b$ miles; in $2\,a$ hours he will travel *twice* as far, that is, $2\,a\,b$ miles.

22. What will y loads of hay come to, at $4\,a$ dollars a load?

23. Multiply $4\,a\,b$ by $x\,y$.
24. Multiply $a\,x$ by $7\,b\,c$.
25. Multiply y by $9\,a\,b\,m$.
26. Multiply $7\,a\,b\,n$ by $x\,y\,z$.
27. Multiply $a\,r\,x$ by $12\,b\,c\,y$.
28. Multiply $9\,a\,m\,n$ by x.
29. Multiply $4\,y$ by $a\,b\,m$.
30. Multiply $17\,a\,m\,x$ by $b\,c\,y\,z$.

31. What will 5 barrels of flour cost, at $2\,a$ dollars per barrel? ANS. \$$10\,a$.

If one barrel cost $2\,a$ dollars, 5 barrels will cost 5 times $2\,a$, that is, $10\,a$ dollars. Let $a = 3$; then $2\,a$ will be twice 3, or 6 dollars a barrel; and 5 barrels will cost $5 \times 6 = 30$ dollars. But $10\,a = 10 \times 3$, or 30 dollars also.

32. A cistern has 4 cocks, each of which will discharge $5\,x$ gallons in an hour. How much will flow from them all in 2 hours? ANS. $40\,x$ gallons.

33. Multiply $8\,a\,x$ by 4.
34. Multiply $12\,a\,b\,x$ by 6.

35. Multiply $9\,a\,b\,c$ by 2.
36. Multiply $15\,a$ by 3.
37. Multiply $8\,x\,y\,z$ by 14.
38. Multiply $12\,b\,c\,m\,y$ by 18.
39. Multiply $73\,b\,m\,y$ by 42.
40 Multiply $19\,a\,b\,x\,z$ by 12.

41. If a man's income be $2\,b$ dollars per day, what will it be in $6\,a$ days? ANS. $12\,a\,b$ dollars.

Here the income of one day must be multiplied by the number of days, and the answer may stand $6\,a\,2\,b$. Let $a = 2$, and $b = 4$; then we shall have $2\,b = 8$, and $6\,a = 12$; and $12 \times 8 = 96$. But $6\,a\,2\,b$, that is, $6 \times 2 \times 2 \times 4$, is also 96. This expression is made more simple, however, if we multiply the two numbers, 2 and 6, together, and use their product $12\,a\,b$, as in the answer given above; for $12\,a\,b$, that is, $12 \times 2 \times 4 = 96$.

And let it be observed, that, although it matters not in what order any two *letters* are written, *two numbers must always be separated, either by a letter, or by the sign* $\times$; for, if $a = 2$ and $b = 4$,

$$6\,a\,2\,b \text{ is } 6 \times 2 \times 2 \times 4 = 96.$$
$$26\,a\,b \text{ is } 26 \times 2 \times 4 = 208.$$
$$62\,a\,b \text{ is } 62 \times 2 \times 4 = 496.$$

From these examples we derive the following RULE for multiplying simple quantities: *Multiply the numbers or coefficients, and annex all the letters in the several quantities to their product.*

42. Multiply $5\,a\,b$ by $6\,x$.
43. Multiply $8\,a\,x$ by $3\,m\,y$.
44. Multiply $19\,x\,y$ by $4\,b\,m$

45. Multiply $24\,x\,y\,z$ by $b\,x$.
46. Multiply $12\,m\,n$ by 52.
47. Multiply $19\,x\,y$ by $13\,a\,b\,z$.
48. Multiply $x\,y$ by $14\,a\,m$.
49. Multiply $71\,a\,m$ by $22\,b\,h\,y$.
50. Multiply $19\,y$ by $12\,a\,x$.

SECTION II.

Compound Quantities.

1. A man gave to his son a dollars, and to his daughter b dollars, every week. How much did he give them both in c weeks? Ans. $a\,c + b\,c$ dollars.

To his son and daughter he gave $a + b$ dollars in one week. Here we have a *compound* quantity, $a + b$, to be multiplied by a *simple* quantity, c, the number of weeks. To the son he gave $a\,c$ dollars, to the daughter $b\,c$ dollars, in c weeks; that is, $a\,c + b\,c$ to the two.

Let $a = 4$, $b = 3$, and $c = 5$; then $a\,c = 20$, $b\,c = 15$, and $a\,c + b\,c = 20 + 15 = 35$. Again, $a + b = 4 + 3 = 7$; and $7 \times 5 = 35$, as before.

Hence, to multiply a compound quantity by a simple one, *we must multiply every term of the former by the latter.*

2.) Multiply $4\,a + 5\,b$ By $3\,c$

$$\frac{4\,a + 5\,b}{\begin{array}{c}3\,c\end{array}}\quad 12\,a\,c + 15\,b\,c$$

3.) $a\,b + 3\,a\,x$ by $5\,a\,c$

$$5\,a\,a\,b\,c + 15\,a\,a\,c\,x$$

4. Multiply $b + b\,c + 5\,x\,y + 13$ by $a\,b$.
5. Multiply $6\,a\,b + 8\,y + 19\,b$ by $4\,a\,x$.
6. Multiply $12\,b + 8\,z + 15$ by $7\,a\,x\,y$.
7. Multiply $11\,a\,b + n + y$ by $9\,a\,m\,x$.
8. Multiply $c\,y + m + 4 + z$ by $a\,b\,x$.
9. Multiply $6\,d\,m + x + 5\,y + 1$ by $7\,a\,m$.
10. Multiply $5\,a\,x\,z + 1 + 3\,x + y$ by $a\,b\,x$.
11. Multiply $7\,n\,x + 5\,y + 8\,a$ by $4\,b\,c\,m$.
12. Multiply $9\,x\,y + 1 + 6\,a\,z$ by a.
13. Multiply $c\,d + b + c + a + x$ by $a\,b$.
14. Multiply $7\,a\,b + 9\,a\,b\,x\,y + 2$ by $c\,d$.

15. What is the product of $a + b$ multiplied by $x + y$? ANS. $a\,x + b\,x + a\,y + b\,y$.

In this question, both the multiplicand and the multiplier are *compound* quantities. Now, if we multiply $a + b$ by x only, the product is $a\,x + b\,x$; but as x is smaller than the whole multiplier by the value of y, we must multiply by this quantity also, and add the product, $a\,y + b\,y$, to the former.

To illustrate this by numbers, let $a = 2$, $b = 3$, $x = 4$, and $y = 5$. Then we have $2 + 3$ to be multiplied by $4 + 5$. Now, $2 + 3 = 5$, and $4 + 5 = 9$, and $5 \times 9 = 45$. Or the work may be expressed thus:

$$
\begin{array}{lll}
2 + 3 & & \\
4 + 5 & & \\
\hline
8 + 12 & & \\
 & 10 + 15 & \\
\hline
\multicolumn{3}{l}{8 + 12 + 10 + 15 = 45.}
\end{array}
$$

16. What is the product of $2a+3b$ multiplied by $3a+2b$?

$$\begin{array}{l} 2a+3b \\ 3a+2b \\ \hline 6aa+9ab \\ \qquad\quad 4ab+6bb \\ \hline \end{array}$$

ANS. $6aa+13ab+6bb$.

The terms $9ab$ and $4ab$ being similar, they are united as in Addition. When both the multiplicand and multiplier are compound quantities, *each term of the former must be multiplied by each term of the latter; and the several products must then be added together.*

17. Multiply $9ay+bd$ by $bd+ay$.
18. Multiply $3abc+8ay+x$ by $abc+x$.
19. Multiply $b+c+8$ by $b+c+10$.
20. Multiply $x+y+z+4$ by $a+c$.
21. Multiply $3ab+c+xy$ by $xy+c+ab$.
22. Multiply $4ab+2b+5a$ by $3a+7b$
23. Multiply $ab+4$ by $5+ab$.

SECTION III.

Signs in Multiplication.

As all the examples hitherto proposed in this Chapter, consist of positive quantities, nothing has been said on the subject of the signs; for it is evident, that

the product of two positive quantities must also be positive. When you multiply $+ a$ by $+ b$, you merely take $+ a$ as many times as there are units in b, without making any alteration in the nature of the quantity expressed by it.

1. A gentleman's income being a dollars, and his expenses b dollars per week, how much will he save in c weeks? Ans. $a c - b c$ dollars.

Here we have $a - b$ to be multiplied by c. His income is $a c$ dollars, and his expenses $b c$ dollars, for the given time; then $a c - b c$ expresses the excess of the former over the latter.

Let $a = 12$, $b = 10$, and $c = 5$; then we shal. have

$$\begin{array}{r} a - b \\ c \\ \hline a c - b c. \end{array} \qquad \begin{array}{r} 12 - 10 = \;\; 2 \\ 5 \quad\;\; 5 \\ \hline 60 - 50 = 10. \end{array}$$

Here, — *multiplied by* +, *gives* — *in the product.*

2. Multiply $3 x - b x$ by $a d$.
3. Multiply $8 a b - 1$ by $a x$.
4. Multiply $9 a x - 17 x$ by $3 a c$.
5. Multiply $a - b x - x y$ by $c m n$.
6. Multiply $a b + x - 1 + a$ by $x y z$.
7. Multiply $3 x y - b x$ by $5 c d$.
8. Multiply $a b c - 1 + 6 x$ by $9 m n$.
9. Multiply $a x - 4 - x$ by $a x + x$.
10. Multiply $b c x - 16$ by $a c + m z$.

11. What is the product of $a + b$ multiplied by $x - y$? Ans. $a x + b x - a y - b y$.

We first multiply $a + b$ by x, which gives $a x + b x$;

but this product is evidently too large, as we are not required to multiply by the *whole* value of x, but only by its *excess* over the value of y. We must, therefore, multiply by y, and subtract the product, $a y + b y$, from the former; but when subtracted, it becomes $- a y - b y$.

Let $a = 7$, $b = 5$, $x = 10$, and $y = 4$; then we shall have

$a + b$	$7 + 5 = 12$
$x - y$	$10 - 4 = 6$
$a x + b x$	$70 + 50 \quad 72$
$- a y - b y$	$- 28 - 20$
$a x + b x - a y - b y.$	$70 + 50 - 28 - 20 = 72.$

In this operation, + *multiplied by* —, *gives* — *in the product.*

12. Multiply $3 a + b y$ by $d - f$.
13. Multiply $a b + c + x$ by $2 c - d$.
14. Multiply $5 a + 2 b$ by $6 a - 3 b$.
15. Multiply $x + y + z$ by $x - y - z$.
16. Multiply $5 x + 5$ by $3 x - 3$.
17. Multiply $a b x + c m$ by $1 - 5 y$.
18. Multiply $r x + a + 16$ by $- 8 x$.
19. Multiply $a + 5 + x$ by $a - x - 5$.
20. Multiply $h m + 16 + y$ by $9 b - 8 b$.

21. Multiply $a - b$ by $x - y$.

Ans. $a x - b x - a y + b y$.

We first multiply $a - b$ by x, and the product is $a x - b x$. But we are not required to multiply $a - b$ by the *whole* value of x, but only by its *excess* over the value of y; we must, therefore, multiply by

y, and subtract this product from the former. The product of $a - b$ multiplied by y, is $a\,y - b\,y$, and when subtracted, it becomes $-a\,y + b\,y$.

Let $a = 15$, $b = 5$, $x = 12$, and $y = 8$; then we have

$$\begin{array}{ll}
a - b & 15 - 5 = 10 \\
x - y & 12 - 8 = 4 \\
\hline
a\,x - b\,x & 180 - 60 \qquad 40 \\
\quad - a\,y + b\,y & \quad - 120 + 40 \\
\hline
a\,x - b\,x - a\,y + b\,y. & 180 - 60 - 120 + 40 = 40.
\end{array}$$

Hence it appears, that $-$ *multiplied by* $-$ *gives* $+$ *in the product.* This is a principle which is very apt to perplex beginners. To use a common expression, they cannot understand how the multiplying of less than nothing by less than nothing, can give a real or positive quantity. Thus stated, the subject is, indeed, quite inexplicable; but the whole difficulty vanishes when it is remembered that there are, in fact, two operations carried on at the same time, namely, Multiplication and Subtraction. We first multiply by the negative quantity, as if it were positive; and then, by changing the signs of all the terms, subtract the product from the quantity already obtained.

22. Multiply $b + a\,c - b\,c$ by $f - c$.
23. Multiply $a - a\,b - 4$ by $b - 5$.
24. Multiply $-6\,a$ by $-6\,a$.
25. Multiply $12\,a\,b - 8$ by $a - 12$.
26. Multiply -8 by -6.
27. Multiply $8\,a\,b - 3\,x$ by $2\,a\,b - 5\,x$

SECTION IV.

General Rule for Multiplication.

In the preceding sections of this Chapter, have been developed all the rules to be observed in the multiplication of algebraic quantities. To facilitate practice, they will now be repeated together.

1. MULTIPLICATION. *Multiply each term of the multiplicand by each term of the multiplier.*

2. SIGNS. *When both terms have the same sign, the product has the sign $+$; but when they have different signs, the product has the sign $-$.*

3. COEFFICIENTS. *Multiply the coefficients of both terms together, and use their product.*

4. LETTERS. *Write the letters of both terms in order, one after the other.*

5. REDUCTION. *Add together the several products by their proper signs, and unite such as are similar into one term.*

1. Multiply $x + 2\,x\,y + y$ by $x - y$.
   ANS. $x\,x + 2\,x\,x\,y - 2\,x\,y\,y - y\,y$.
2. Multiply $x + x\,y + y$ by $x - x\,y + y$.
3. Multiply $3\,x - 2\,x\,y + 5$ by $x + 2\,x\,y - 3$.
4. Multiply $2\,a - 3\,a\,x + 4\,x$ by $5\,a - 6\,a\,x - 2\,x$.
5. Multiply $x + 3\,b - 6$ by $4\,x - 8\,b - 8$.
6. What is the product of $7\,l - 2\,m - 9$ multiplied by $3\,l - 11\,m$?
7. Multiply $a + b + 6$ by $a - b - 6$.
8. Multiply $a\,b - b\,c + c\,d$ by $a\,b + b\,c - c\,d$.

9. What is the product of $5 + b - c$ multiplied by $a - b + 9$?

10. What is the product of $7a - 3x + 5$ multiplied by $4a + d$?

11. Multiply $bb - bc + cc$ by $b + c$.

12. Required the product of $3mmx - mx + x$ multiplied by $2m - y$.

13. What is the product of $xx + xy + 3z$ multiplied by $x + 1$?

14. Multiply $a + b - c$ by $1 - x$.

15. What is the product of $aa - 4ax + xx$ multiplied by $a + x$?

16. Required the product of $aa - 2ay + yy$ multiplied by $a - y$.

17. Multiply $a + c - g$ by $4aa + 3cg$.

18. What is the product of $m - n + z$ multiplied by $5mn - 2yz$?

19. Required the product of $16am + 3ac$ multiplied by $-ay + 4m$.

20. Multiply $aa - bb + 8$ by $x - yyy$.

21. What is the product of $xx + yyy - 7$ multiplied by $x - y$?

22. Required the product of $a + b + c + d$ multiplied by $a - b - c - d$.

23. Multiply $13xy - 12x$ by $y - 1$.

24. Multiply $3a + 5 - 2b$ by $5a + 2b - 5$.

CHAPTER V.

DIVISION.

SECTION I.

Simple Quantities.

1. If you divide 15 cents equally among 3 boys, how many cents will each boy have? Ans. 5 cents.

When any given quantity is to be separated into a certain number of equal parts, the value of one of those parts is determined by *Division*. Thus, if we were to count off 15 cents into three equal piles, we should find that each of those piles would contain 5 cents; that is, 3 is contained 5 times in 15.

The quantity to be divided is called the *Dividend*; the quantity, denoting the number of equal parts into which the dividend is to be divided, is called the *Divisor*; and the value of one of those parts is called the *Quotient*. Thus, in the above example,

15 is the Dividend,
3 is the Divisor,
5 is the Quotient.

As Division is the reverse of Multiplication, *the divisor and quotient being multiplied together, will*

reproduce the dividend. Thus, in the question above, $3 \times 5 = 15$. Indeed, the dividend may be regarded as a product, of which one of the factors, the divisor, is known; and the whole object of Division is, to find the other factor, namely, the quotient.

2. $6\,a \div a = 6$; for $6 \times a = 6\,a$.
3. $a\,b \div b = a$; for $a \times b = a\,b$.
4. $a\,b\,c \div a = b\,c$; for $b\,c \times a = a\,b\,c$.
5. $a\,b\,c \div b = a\,c$; for $a\,c \times b = a\,b\,c$.
6. $a\,b\,c \div c = a\,b$; for $a\,b \times c = a\,b\,c$.
7. $16\,a \div 4 = 4\,a$; for $4\,a \times 4 = 16\,a$.
8. $20\,a\,b \div 5\,a = 4\,b$; for $4\,b \times 5\,a = 20\,a\,b$.
9. $32\,a\,b\,c \div 8\,a\,b = 4\,c$; for $4\,c \times 8\,a\,b = 32\,a\,b\,c$.

In each of these examples, our object has been, to find some quantity, which, being multiplied by the divisor, would produce the dividend. The mode in which this may be done, is evident, namely: *Divide the coefficient of the dividend by that of the divisor; and omit in the quotient all those letters which are common to the given terms.*

10. Divide $18\,a\,b\,m\,x$ by $9\,a\,b\,x$.
11. Divide $27\,a\,b\,c\,m\,y$ by $3\,b\,c\,y$.
12. Divide $9\,h\,m\,n\,p$ by $3\,h\,m$.
13. Divide $a\,b\,x$ by $a\,b\,x$.
14. Divide $39\,a\,z$ by $13\,z$.
15. Divide $108\,a\,b\,c\,h\,m\,n\,x$ by $12\,a\,c\,m\,x$
16. Divide $24\,a\,x\,y\,z$ by $12\,x\,y\,z$.
17. Divide $36\,a\,x\,y$ by $9\,a\,x\,y$.
18. Divide $72\,a\,b\,c\,x\,y$ by $8\,a\,b\,x$.
19. Divide $18\,h\,m\,n\,x\,z$ by $6\,h\,m\,x\,z$.

20. Divide $56\,a\,x\,y\,z$ by $7\,x\,y\,z$.

21. Divide a by b. Ans. $\frac{a}{b}$.

Here, the given quantities being dissimilar, the division cannot be performed, but only represented; we therefore *write the divisor under the dividend, in the form of a fraction.*

22. Divide x by y.

23. Divide $a\,b$ by $c\,x$.

24. Divide $4\,a\,b\,c$ by $5\,x\,y$.

25. Divide $17\,a\,b\,x$ by $21\,c\,d\,y$.

26. Divide $a\,b$ by $a\,x$. Ans. $\frac{b}{x}$.

This may be written $\frac{ab}{ax}$, and then reduced, as in the answer, the two a's being cancelled. Let $a = 4$, $b = 5$, and $x = 6$; then $a\,b = 4 \times 5$, and also $a\,x = 4 \times 6$; and $\frac{ab}{ax} = \frac{4 \times 5}{4 \times 6} = \frac{20}{24} = \frac{5}{6}$, as above.

27. Divide $a\,b\,c$ by $a\,b\,x$.

28. Divide $a\,b\,m\,x$ by $a\,b\,x\,y$.

29. Divide $a\,b\,m$ by $a\,b\,m$.

30. Divide $a\,b\,m\,x$ by $a\,b\,m\,x\,y$.

31. Divide $a\,b\,x\,y$ by $a\,b\,x$.

32. Divide $b\,c\,m\,x\,y$ by $a\,b\,c\,x\,z$.

33. Divide $14\,a\,b$ by $7\,c$.

We may first express the division thus, $\frac{14\,a\,b}{7\,c}$, and reduce the result by dividing both terms by 7.

34. Divide $16\,a\,b$ by $4\,x$.

35. Divide $20\,a\,b$ by $4\,a\,x$.

36. Divide $15\,x\,y$ by $5\,a\,x\,y$.

37. Divide $25\,a\,b\,c\,m$ by $10\,a\,b\,x$.

38. Divide $18\,a\,b\,f\,m$ by $6\,a\,b\,f\,m\,n\,x$.
39. Divide $21\,a\,x$ by $7\,a\,x\,y$.
40. Divide $5\,g\,h$ by $3\,x\,y$.
41. Divide $14\,a\,b\,x$ by $2\,a\,b\,x$.
42. Divide $a\,b\,f\,g\,h$ by $a\,f\,h\,x$.
43. Divide $49\,a\,b\,c\,x\,y$ by $7\,a\,b\,c\,d\,m\,x$.
44. Divide $a\,b\,m\,x$ by $4\,a\,b\,m\,y$.
45. Divide $12\,(a - b + 9)$ by $8\,(a - b + 9)$.
46. What is the quotient of $a\,b\,(19 + x - y)$ divided by $b\,(19 + x - y)$?
47. Divide $12\,a\,b\,(x - y + z)$ by $3\,a\,(x - y + z)$
48. Divide $4\,(a\,b - 10 + x)$ by $7\,(a\,b - 10 + x)$.
49. Divide $3\,(a\,y - 12\,b)$ by $a\,(a\,y - 12\,b)$.

SECTION II.

Signs in Division.

The process of dividing one algebraic quantity by another, consists of two parts: the first is, to ascertain the proper expression of the quotient, in letters and figures; the other is, to determine the character of that expression, either as positive or negative. In the last section, the mode of *dividing* one simple quantity by another, was considered alone, without any regard to the *signs*. The mode of determining these, will form the subject of the present section.

1. Divide $a\,b$ by b. Ans. a.

In this example, both of the given quantities are *positive*. And the divisor, b, being $+$, the quotient,

a, must be $+$ also; for, multiplied together, they must give the product $+ a b$, that is, the dividend. But if we suppose the quotient to be $- a$, we shall have $- a \times + b$, which will give $- a b$. Hence, $+$ *divided by* $+$ *gives* $+$ *in the quotient.*

2. Divide $a b$ by $- b$. Ans. $- a$.

Here, the divisor, b, being $-$, the quotient, a, must be $-$ also; for $+ a \times - b$ will give $- a b$, and not $+ a b$, which is the dividend. Hence, $+$ *divided by* $-$, *gives* $-$ *in the quotient.*

3. Divide $16 a b x$ by $- 8 a x$. Ans. $- 2 b$.
4. Divide $36 a h m n y$ by $- 12 a m n$.
5. Divide $72 a b x y$ by $- 9 a x y$.
6. Divide $a b x$ by $- a b x$.
7. Divide $12 a m n$ by $- 4 a m n$.
8. Divide $16 a b r z$ by $- 8 a b r$.
9. Divide $28 a b m n p$ by $- 7 a m p$.
10. Divide $48 a h y$ by $- 16 a y$.

11. Divide $- a b$ by b. Ans. $- a$.

In this example, as the dividend, $a b$, is a $-$ quantity, and the divisor, b, is $+$, the quotient must be $-$; for $- a \times + b$ gives $- a b$, the dividend. Hence, $-$ *divided by* $+$, *gives* $-$ *in the quotient.*

12. Divide $- 16 a x$ by $4 a x$. Ans. $- 4$.
13. Divide $- 21 a b m n$ by $7 a m n$.
14. Divide $- 35 x y z$ by $7 x z$.
15. Divide $- 32 a b c m y$ by $16 a b m y$.
16. Divide $- a b$ by $a b$.
17. Divide $- 28$ by 7.
18. Divide $- 49 a x z$ by $7 a z$.
19. Divide $- 96 a b c x$ by $12 c x$.

20. Divide $-28\,a\,m\,n\,p$ by $7\,a\,m\,n$.

21. Divide $-a\,b$ by $-b$. ANS. a.

Here, as the divisor is $-$, the quotient must be $+$; for their product must produce the dividend, namely, $-a\,b$, which is the product of $+a \times -b$. But if we suppose the quotient to be $-$, then we shall have $-a \times -b = +a\,b$, which is evidently wrong. *Hence, $-$ divided by $-$ gives $+$ in the quotient.*

22. Divide $-18\,a\,b\,m\,x$ by $-6\,a\,b\,m$. ANS. $3\,x$.
23. Divide $-25\,h\,m\,n\,x$ by $-5\,h\,m$.
24. Divide $-32\,a\,b\,c\,y$ by $-8\,a\,c$.
25. Divide $-45\,h\,m\,r\,x$ by $-9\,m\,x$.
26. Divide $-12\,b\,c$ by $-3\,c$.
27. Divide $-b\,c\,x$ by $-b\,c$.
28. Divide $-27\,y\,z$ by $-9\,y$.
29. Divide $-64\,m\,n$ by $-16\,m$.
30. Divide $-9\,x\,y\,z$ by $-3\,x\,y\,z$.

From the preceding examples and remarks, we derive the following general RULE for the signs in Division:

When the signs of the divisor and dividend are alike, the sign of the quotient is $+$; when they are not alike, it is $-$

31. Divide $18\,a\,b\,x\,y$ by $-9\,a\,x$. ANS. $-2\,b\,y$.
32. Divide $-64\,a\,m\,x$ by $8\,a\,m$.
33. Divide $81\,a\,b\,c\,m\,n$ by $9\,a\,b\,c$.
34. Divide $-63\,h\,m\,n\,p$ by $-7\,h\,m$.
35. Divide $84\,a\,y$ by $12\,a\,y$.
36. Divide $-21\,a\,b\,x\,y\,z$ by $-9\,a\,b\,x$.
37. Divide $42\,h\,m\,x$ by $-7\,x$.

38. Divide $a\,b$ by $c\,d$.
39. Divide a by $-\,b$.
40. Divide $-\,9\,a\,x$ by $4\,x$.
41. Divide $9\,x\,y$ by $-\,4\,y$.
42. Divide $-\,12\,b\,c\,d$ by $-\,3\,b\,c\,x$.
43. Divide $18\,x\,z$ by $-\,6\,x\,y$.
44. Divide $7\,m\,n$ by $-\,3\,n$.
45. Divide $-\,2\,x\,z$ by $3\,z$.
46. Divide $-\,7\,a\,b$ by $-\,3\,a$.
47. Divide $7\,a\,b$ by $3\,a$.
48. Divide $-\,8\,a\,b\,c$ by $3\,a\,b\,c$.

SECTION III.

Compound Quantities.

1. Divide $a\,b + a\,c$ by a. Ans. $b + c$.

We have here a *compound* quantity, $a\,b + a\,c$, to be divided by a *simple* quantity, a. We first divide $a\,b$ by a, and the quotient is b; we then divide $a\,c$ by a, and the quotient is c: that is, *we divide each term of the compound quantity separately.*

2. Divide $12\,a\,c + 9\,b\,c$ by $3\,c$. Ans. $4\,a + 3\,b$.
3. Divide $18\,a\,x + 15 - 21\,a\,b$ by $3\,a$.
4. Divide $3\,a\,b\,c + 12\,a\,b\,x - 9\,a\,b$ by $3\,a\,b$.
5. Divide $10\,a\,x - 15\,x$ by $5\,x$.
6. Divide $a\,r\,x + a\,h\,x - a$ by $a\,x$.
7. Divide $a\,x - a\,y + a\,z$ by a.
8 Divide $6\,a\,b + 12\,a\,c$ by $3\,a$.

9. Divide $35\,d\,n + 14\,d\,x$ by $7\,d$.
10. Divide $12\,a\,b\,y + 6\,a\,b\,x - 18\,b\,m + 24\,b$ by $6\,b$.
11. Divide $16\,a - 8 + 12\,y - 20\,a\,d\,x + m$ by 4.
12. Divide $15\,a\,b - 25\,b\,x + 10\,a\,m$ by $10\,a$.
13. Divide $28\,a\,b\,x - 14\,b\,x - 49\,b\,x\,y$ by $7\,b\,x$.
14. Divide $16\,a\,b - 12\,a\,c$ by $4\,a$.
15. Divide $12\,x\,y - 18\,y$ by $6\,x$.
16. Divide $24\,a\,b - 24\,a$ by $8\,a$.
17. Divide $21\,a\,b - 14\,x$ by $-7\,a$.
18. Divide $a + b - 8$ by a.
19. Divide $15\,m\,n + 35\,x$ by $5\,x$.
20. Divide $17 - 12\,a + a\,b$ by $3\,a$.
21. Divide $a + b$ by $x + y$. ANS. $\frac{a+b}{x+y}$.

In this example, both the divisor and dividend are *compound* quantities; but, as all their terms are dissimilar, it is evident that the division can only be represented, and not actually performed.

22. Divide $a\,c + b\,c$ by $a + b$.

$$\begin{array}{r l}
a + b\,)\; & a\,c + b\,c\;(\,c \\
 & \underline{a\,c + b\,c} \\
 & \;\;*\qquad *
\end{array}$$

In this example, we divide the first term of the dividend, $a\,c$, by the first term of the divisor, a, and obtain c for the quotient. We then multiply the whole divisor, $a + b$, by c, to ascertain whether it be the whole quotient, or only a part of it; and the product is $a\,c + b\,c$, that is, the dividend.

23. Divide $b\,x + c\,x$ by $b + c$.
24. Divide $a\,a + a\,b$ by $a + b$.

25. Divide $3\,a\,a - 2\,a\,b$ by $3\,a - 2\,b$.

26. Divide $12\,a\,a + 6\,a\,b$ by $4\,a + 2\,b$.

27. Divide $18\,a\,a\,a + 6\,a\,a\,b - 12\,a\,a\,x$ by $6\,a + 2\,b - 4\,x$.

28. Divide $14\,a\,b + 21\,a\,c + 49\,a\,x$ by $2\,b + 3\,c + 7\,x$.

29. Divide $b\,b + 3\,b\,c + 2\,c\,c$ by $b + c$.

$$
\begin{array}{r|l}
b + c\,) & b\,b + 3\,b\,c + 2\,c\,c\ (\,b + 2\,c. \\
 & \underline{b\,b + \,b\,c} \\
 & \ 2\,b\,c + 2\,c\,c \\
 & \ \underline{2\,b\,c + 2\,c\,c} \\
 & \ \ \ *\qquad *
\end{array}
$$

In this example, as in the preceding, we divide the first term of the dividend by the first term of the divisor. The quotient of $b\,b$ divided by b, is b. We then multiply the whole divisor, $b + c$, by b, and obtain the product $b\,b + b\,c$, which is not equal to the dividend. We are, therefore, certain that b is not the *whole* quotient. This product is then subtracted from the dividend, and the remainder is $2\,b\,c + 2\,c\,c$, which must also be divided. We begin as before, and divide the first term of this remainder by the first term of the divisor; that is, $\frac{2\,b\,c}{b} = 2\,c$, which is the second term of the quotient. We next multiply the divisor, $b + c$, by $2\,c$, and the product is equal to the remainder of the dividend. The whole quotient is $b + 2\,c$, which, being multiplied by the divisor, will reproduce the dividend.

30. Divide $a\,a + 2\,a\,b + b\,b$ by $a + b$.

$$
\begin{array}{l}
a + b\;)\; a\,a + 2\,a\,b + b\,b\; (\,a + b. \\
 \underline{a\,a + \,a\,b} \\
 a\,b + b\,b \\
 \underline{a\,b + b\,b} \\
 \;*\quad\;\; *
\end{array}
$$

31. Divide $a\,a - b\,b$ by $a + b$.

$$
\begin{array}{l}
a + b\;)\; a\,a - b\,b\; (\,a - b. \\
 \underline{a\,a + a\,b} \\
 - a\,b - b\,b \\
 \underline{- a\,b - b\,b} \\
 *\quad\;\; *
\end{array}
$$

This example is like the former, excepting that it contains negative quantities. We obtain the second dividend, $- a\,b - b\,b$, by subtracting $a\,a + a\,b$ from $a\,a - b\,b$; and the sign of the second term of the quotient must be $-$, because the signs of the divisor, $+ a$, and of the dividend, $- a\,b$, are not alike. It is obvious that the first term of the quotient is too large; for $(a + b)\,a = a\,a + a\,b$, which is evidently a larger quantity than the dividend, $a\,a - b\,b$, whatever may be the values of a and b.

32. Divide $8\,x\,x\,x - y\,y\,y$ by $2\,x - y$.

$$
\begin{array}{l}
2x - y\,)\; 8\,x\,x\,x - y\,y\,y\; (\,4\,x\,x + 2\,x\,y + y\,y. \\
 \underline{8\,x\,x\,x - 4\,x\,x\,y} \\
 4\,x\,x\,y - y\,y\,y \\
 \underline{4\,x\,x\,y - 2\,x\,y\,y} \\
 2\,x\,y\,y - y\,y\,y \\
 \underline{2\,x\,y\,y - y\,y\,y} \\
 *\qquad\quad *
\end{array}
$$

SECTION IV.

General Rule for Division.

From the foregoing examples and observations, we derive the following general RULE for Division in Algebra, when both the divisor and dividend are compound quantities:

Divide the first term of the dividend by the first term of the divisor, for the first term of the quotient.

Multiply the whole divisor by this term, and subtract the product from the dividend.

Divide the first term of the remainder by the first term of the divisor, for the second term of the quotient.

Multiply the whole divisor by this second term, and subtract the product from the remainder.

Continue this series of operations as long as the nature of the question may require.

1. Divide $a\,a + a\,b + a\,c + 5\,a + 5\,b + 5\,c$ by $a + b + c$.

2. Divide $a\,a - 6\,a + a\,b - a\,c + 6\,c - b\,c$ by $a - c$.

3. Divide $b\,b\,b + c\,c\,c$ by $b + c$.

4. Divide $6\,m\,m\,m\,x - 3\,m\,m\,x\,y - 2\,m\,m\,x + m\,x\,y + 2\,m\,x - x\,y$ by $2\,m - y$.

5. Divide $x\,x\,x + x\,x + x\,x\,y + x\,y + 3\,x\,z + 3\,z$ by $x + 1$.

6. Divide $a + b - c - a\,x - b\,x + c\,x$ by $a + b - c$.

7. Divide $a\,a\,a - 3\,a\,a\,x - 3\,a\,x\,x + x\,x\,x$ by $a + x$.

8. Divide $a\,a\,a - 3\,a\,a\,y + 3\,a\,y\,y - y\,y\,y$ by $a - y$.

9. Divide $4\,a\,a\,a + 3\,a\,c\,g + 4\,a\,a\,c + 3\,c\,c\,g - 4\,a\,a\,g - 3\,c\,g\,g$ by $a + c - g$.

10. Divide $5\,m\,m\,n - 2\,m\,y\,z - 5\,m\,n\,n + 2\,n\,y\,z + 5\,m\,n\,z - 2\,y\,z\,z$ by $m - n + z$.

11. Divide $x\,x\,x - 3\,x\,x\,y + 4\,x\,y\,y - 4\,y\,y\,y$ by $x\,x - 2\,x\,y + y\,y$.

$$
\begin{array}{r|l}
xx-2xy+yy) & xxx-3xxy+4xyy-4yyy\left(x-y+\dfrac{xyy-3yyy}{xx-2xy+yy}\right. \\
 & \underline{xxx-2xxy+xyy} \\
 & -xxy+3xyy-4yyy \\
 & \underline{-xxy+2xyy-yyy} \\
 & xyy-3yyy
\end{array}
$$

As there is a remainder in this example, it is annexed to the quotient in the form of a fraction.

12. Divide $a\,x + b\,x + a\,y + b\,y + z$ by $a + b$.

13. Divide $c\,d - c\,h + d\,e - e\,h + x$ by $d - h$.

14. Divide $x\,x\,x\,y - x\,x\,y\,y + 5\,y\,y\,y$ by $x - y$.

15. Divide $3\,a\,b\,b\,b + 2\,a\,a\,b\,b\,c - a\,b\,b + b\,b\,b\,c + b\,b\,x - 3\,a\,a\,b - 2\,a\,a\,a\,c + a\,a - a\,b\,c - a\,x$, by $3\,a\,b + 2\,a\,a\,c - a + b\,c + x$.

16. Divide $18\,a\,a\,b + 3\,a\,a\,c\,c - 3\,a\,x\,x + 3\,a\,b\,b\,c - 6\,a\,b\,b - a\,b\,c\,c + b\,x\,x - b\,b\,b\,c + a\,x + c$, by $6\,a\,b + a\,c\,c - x\,x + b\,b\,c$.

17. Divide $-16\,a\,a\,m\,y - 3\,a\,a\,c\,y + 64\,a\,m\,m + 12\,a\,c\,m$ by $16\,a\,m + 3\,a\,c$.

18. Divide $12\,a\,a\,b - 8\,a - 144\,a\,b + 96$ by $a - 12$.

CHAPTER VI.

FRACTIONS.

SECTION I.

Introduction and Definitions.

1. Let it be required to distribute a dollars equally among b poor persons. What will be the share of each? Ans. \$ $\frac{a}{b}$.

The number of dollars must be divided by the number of persons; but, as the division cannot be actually performed, all that can be done is, to *represent* it as above. So, too, if we suppose $a = 3$, and $b = 5$, we must indicate the answer in a similar manner; thus, $\frac{3}{5}$.

These expressions, $\frac{a}{b}$, $\frac{3}{5}$, and others like them, are called *Fractions*, from a Latin word, which signifies *broken*, because *the value of a fraction is always expressed in parts of a whole one.* If, for instance, we cut an apple into 5 equal parts, and give 3 of those parts to a boy, he will have $\frac{3}{5}$ of the apple, which expression is read three fifths. Other fractions are read in a similar manner; as, $\frac{1}{2}$, one half; $\frac{2}{3}$, two thirds; $\frac{3}{4}$, three fourths; $\frac{5}{6}$, five sixths; $\frac{7}{8}$, seven eighths, &c.

The number below the line, which shows into how many parts the unit or quantity is divided, is called the Denominator.

The number above the line, which shows how many of the parts are taken, is called the Numerator.

In the fractions $\frac{a}{b}$, $\frac{m}{n}$, $\frac{6}{7}$, $\frac{19}{4}$, $\frac{a+b}{x-y}$, and $\frac{3}{z}$, the numerators are a, m, 6, 19, $a+b$, and 3; and the denominators are b, n, 7, 4, $x-y$, and z.

A Proper fraction is one whose value is less than a unit; that is, whose numerator is less than its denominator; as, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{7}{12}$, $\frac{a}{a+b}$, $\frac{x-y}{x+y}$, &c.

An Improper fraction is either equal to or greater than a unit; as, $\frac{2}{2}$, $\frac{14}{5}$, $\frac{49}{7}$, $\frac{a}{a}$, $\frac{ab}{a}$, $\frac{ab+b}{a}$, &c. It is evident that the division here represented can be performed, either wholly or in part.

Expressions consisting of a whole number and a fraction, are called Mixed numbers; as, $2\frac{4}{5}$, and $b+\frac{b}{a}$, and 2 and b are called integers, or integral quantities. It often happens, both in Arithmetic and Algebra, that there is a remainder after division, which should be written above the divisor, and annexed to the quotient in the form of a fraction. Hence the origin of mixed numbers. [See Chap. V. Sec. III. and IV.]

Since fractions always imply division, *any quotient may be expressed in the form of a fraction, the dividend being the numerator, and the divisor the denominator.*

Express the answers of the following examples in the form of fractions:

2. Divide 476 by 19. ANS. $\frac{476}{19}$.

3. Divide $6\,a\,b\,c$ by d.

4. Divide $8\,a + b - x$ by $c - d$.

5. Divide $17\,x\,y - m + 7$ by $x + 10$.

6. Divide $a - m + 7 - 5\,y$ by $10\,b - x + 14$.

As the value of any quantity is not altered when it is divided by a unit, *we can convert an integer into a fraction by making* 1 *the denominator.*

Convert the following quantities into fractions:

7.) $8 = \frac{8}{1}$. 8.) $a = \frac{a}{1}$. 9.) $a + b = \frac{a+b}{1}$

10.) 24 11.) $a - b$. 12.) $a + b - m\,n$.

The numerator of a fraction being a dividend, and the denominator a divisor, it follows, that when these are equal to each other, as $\frac{2}{2}$, the *value* of the fraction is 1; for 2 divided by 2 gives 1. Therefore $\frac{1}{1}$, $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, $\frac{5}{5}$, $\frac{a}{a}$, and $\frac{a\,b}{a\,b}$, are all equal to each other, the value of each fraction being unity, or 1. Hence, it is evident, that if *both the numerator and denominator be either multiplied or divided by the same number or letter, the value of the fraction is not changed.* On this principle are founded the rules for bringing fractions to a common denominator, and for reducing them to their least terms.

13. Express 6 in the form of a fraction, having 4 for its denominator. ANS. $\frac{24}{4}$.

It is evident that the numerator must be 4 times

6; for, when divided by 4, the denominator, the quotient must be 6.

14. Given the quantity x to be expressed in the form of a fraction, having y for a denominator.

Ans. $\frac{xy}{y}$.

We may first convert x into this fraction, $\frac{x}{1}$, and then multiply both terms by y, according to the principles already given. In other words, to change a whole quantity to a fraction, having a given denominator, *we multiply the whole quantity by the given denominator; and the product is the numerator of the fraction required.*

15. Change 8 to a fraction, having 5 for its denominator.

16. Change $a + b$ to a fraction, having c for its denominator.

17. Express $x + y$ in the form of a fraction, having $5z$ for its denominator.

18. Change $a - 5$ to a fraction, having $2 + b$ for its denominator.

19. Change $5a - bc$ to a fraction, whose denominator shall be $4x$.

20. Express $x + b - 5$ in the form of a fraction, having $3a - z$ for the denominator.

21. Change $5a - 2 + y$ to a fraction, having 6 for its denominator.

22. Change $28 - x + 4y$ to a fraction, whose denominator shall be $2a$.

23. Express $2ab + x - 5$ in the form of a fraction, having $5ab$ for its denominator.

SECTION II.

Reduction.

It is often convenient, and sometimes necessary, to change the form in which a fraction or a mixed quantity is expressed. For instance, to prepare different fractions for addition or subtraction, we must always express them in other fractions, which shall have a common denominator. The process, by which the *form* of a fraction is changed, its *value* remaining the same, is called Reduction.

A. *To reduce an Improper Fraction to a Mixed Quantity.*

1. What is the value of $\frac{8}{8}$? Ans. 1.

2. What is the value of $\frac{19}{5}$? Ans. $3\frac{4}{5}$.

As 5 fifths are equal to 1, 19 fifths must be equal to 3, and 4 fifths more.

3. What is the value of $\frac{ab}{a}$? Ans. b.

4. What is the value of $\frac{ab+b}{a}$? Ans. $b+\frac{b}{a}$

To reduce an improper fraction to a whole or mixed quantity, *Divide the numerator by the denominator, and annex the remainder, if any, to the quotient, in the form of a fraction.*

Reduce the following fractions to whole or mixed quantities:

5.) $\frac{17}{9}$. 6.) $\frac{24}{3}$. 7.) $\frac{39}{5}$. 8.) $\frac{127}{8}$.

9.) $\frac{a}{a}$. 10.) $\frac{5ab}{b}$. 11.) $\frac{a+b}{a+b}$. 12.) $\frac{18xy}{6x}$.

13. What is the value of $\frac{4ab-bb}{a}$?

14. What is the value of $\frac{aaa-bbb}{a-b}$?

15. Reduce $\frac{a+b}{a-b}$ to a mixed quantity.

B. *To reduce a Mixed Quantity to an Improper Fraction.*

16. Reduce $7\frac{2}{3}$ to an improper fraction.

Ans. $\frac{23}{3}$.

In 1 there are 3 thirds, and in 7 there are 7 times 3, or 21 thirds; and 2 thirds added to 21 make 23 thirds.

17. Reduce $b+\frac{b}{a}$ to an improper fraction.

Ans. $\frac{ab+b}{a}$.

The integral quantity, b, reduced to a fraction, becomes $\frac{ab}{a}$; [See Sec. I.] and the whole quantity is $\frac{ab}{a}+\frac{b}{a}$, or $\frac{ab+b}{a}$, which expresses the same value.

Hence, to reduce a mixed number to an improper fraction, *multiply the integer by the denominator, and add the numerator to the product. The sum will be the numerator of the fraction required.*

Reduce the following mixed numbers to improper fractions:

18) $9\frac{4}{7}$. 19.) $a+\frac{a}{b}$. 20.) $xy+10+\frac{x}{y}$.

21.) $b+\frac{a}{a+b}$. 22.) $a-x+\frac{a}{a-x}$. 23.) $aa+\frac{bb}{aa}$.

42.) $\frac{ax+ay}{ab+ax}=\frac{x+y}{b+x}$ 43.) $\frac{13a}{39x}$.

44.) $\frac{3xy}{9ax+27xy}$. 45.) $\frac{2aab-4aax}{6aaa+10aay}$.

46.) $\frac{24aa-12}{16ay+20}$. 47.) $\frac{8axy-4ay+16aay}{12axy-20ay+24axyz}$.

48. Reduce $\frac{ax+by-xy}{ay+cy+ax}$ to its least terms.

49. Reduce $\frac{5ab-10ax+15ay}{20az-10am+5ab}$ to its least terms

SECTION III

The Signs of Fractions.

The learner will remember, that the numerator is a dividend, that the denominator is a divisor, and that the *value* of the fraction is to be found in the quotient. The signs prefixed to the several terms of a fraction, affect those terms only; but when a sign is prefixed to the fraction itself, it affects its *whole* value, that is, the value of all the terms taken collectively. Without regarding the signs, we know that the value of the fraction $\frac{ab}{a}$ is b. Now, by the common rules of division, we have the following cases:

$$\frac{+ab}{+a}=+b,$$

$$\frac{-ab}{-a}=+b,$$

$$\frac{-ab}{+a}=-b,$$

$$\frac{+ab}{-a}=-b.$$

Whence we infer, that *the signs of both the numerator and the denominator may be changed without altering the value of the fraction; and that the value is altered when the sign of either is changed, and not that of the other.*

1. Let it be required to reduce $7\,b - \frac{c-d}{2\,a}$ to an improper fraction. ANS. $\frac{14\,a\,b - c + d}{2\,a}$.

In this example, the value of the fraction $\frac{c-d}{2\,a}$ is to be *subtracted* from the integral part, $7\,b$. For the sign $-$, prefixed to the fraction, does not apply to the first term of the numerator, nor to the denominator, but to the value of the whole fraction. If we reduce the integral part to the form of a fraction, we shall have $\frac{14\,a\,b}{2\,a} - \frac{c-d}{2\,a}$. Let us now reject the denominator, $2\,a$, and subtract the numerator, $c - d$, from $14\,a\,b$, as if they were integral quantities; and we shall obtain $14\,a\,b - c + d$, by the rules for subtraction. It only remains to restore the common denominator, and we have the answer given above.

2. Reduce $7\,b - \frac{c+d}{2\,a}$ to an improper fraction.

ANS. $\frac{14\,a\,b - c - d}{2\,a}$.

Here the quantity $c + d$ is subtracted from $14\,a\,b$, and both signs are changed, as before.

3. Let $7\,b + \frac{c-d}{2\,a}$ be reduced to an improper fraction. ANS. $\frac{14\,a\,b + c - d}{2\,a}$.

4. Let $7\,b + \frac{c+d}{2\,a}$ be reduced to an improper fraction. ANS. $\frac{14\,a\,b+c+d}{2\,a}$.

In the last two examples, as the value of the fraction $\frac{c-d}{2\,a}$ in one, and of $\frac{c+d}{2\,a}$ in the other, is to be *added* to $7\,b$, none of the signs are changed.

5. Reduce $3\ a - \frac{a+b}{c}$ to an improper fraction.

6. Reduce $x\,y - \frac{b-7}{8}$ to an improper fraction.

7. Reduce $a\,b + \frac{14-y}{3\,m}$ to an improper fraction.

8. Reduce $8\,b\,x + \frac{5\,m+7\,y}{4\,a}$ to an improper fraction.

9. Reduce $9\,m - \frac{6\,x-12\,y}{3\,a}$ to an improper fraction.

10. Reduce $x + \frac{a-b}{x-y}$ to an improper fraction.

SECTION IV.

Addition.

1. What is the sum of $\frac{2a}{c}$ and $\frac{3a}{c}$? ANS. $\frac{5a}{c}$.

As these fractions have a common denominator, we evidently obtain the true answer by adding together their numerators, which are similar quantities; and $3\,a + 2\,a = 5\,a$. For, if $a = 4$, and $c = 2$, then $\frac{2\,a}{c} = \frac{8}{2}$, or 4; and $\frac{3\,a}{c} = \frac{12}{2}$, or 6; and $4 + 6 = 10$: but $\frac{5\,a}{c} = \frac{20}{2}$, or 10, also

2. Add together $\frac{3a}{c}$, $\frac{5a}{c}$, $\frac{6a}{c}$ and $\frac{a}{c}$

3. What is the sum of $\frac{a+b}{x-y}$, $\frac{3a+2b}{x-y}$ and $\frac{4a-6b}{x-y}$?

4. Add together $\frac{x-y+z}{2ab}$, $\frac{2y-z-3x}{2ab}$ and $\frac{2x-y+z}{2ab}$.

5. What is the sum of $\frac{a}{c}$ and $\frac{b}{c}$? ANS. $\frac{a+b}{c}$.

Although these fractions have a common denominator, still, as the numerators are dissimilar, they can only be added by means of the sign $+$. Let $a = 4$, $b = 6$, and $c = 2$; then $\left(\frac{a}{c} + \frac{b}{c}\right) = \left(\frac{4}{2} + \frac{6}{2}\right) = 2 + 3$, or 5; also $\frac{a+b}{c} = \frac{4+6}{2} = \frac{10}{2}$, or 5.

6. What is the sum of $\frac{a}{x}$, $\frac{b}{x}$, $\frac{c}{x}$ and $\frac{2d}{x}$?

7. Add together $\frac{2a}{x-y}$, $\frac{3b}{x-y}$, $\frac{3a}{x-y}$ and $\frac{c}{x-y}$.

8. Add together $\frac{3a-b}{x+7-z}$, $\frac{c+2b}{x+7-z}$ and $\frac{a-c-b}{x+7-z}$.

9. What is the sum of $\frac{a-7+2c}{x+y}$, $\frac{3b+18+c}{x+y}$ and $\frac{a-4b-10-3c}{x+y}$?

10. What is the sum of $\frac{a}{b}$ and $\frac{c}{d}$? ANS. $\frac{ad+bc}{bd}$.

These fractions having different denominators, they can only be added, as they are proposed, by means of the sign $+$; we, therefore, reduce them to a common denominator, and proceed as before.

From these examples and observations, we derive the following RULE for adding algebraic fractions: *Reduce the given fractions to a common denominator;*

add their numerators together; and place the sum, for a new numerator, over the common denominator.

11. Add together $\frac{5a}{bc}$, $\frac{3c}{bc}$ and $\frac{4b}{cd}$.

$$\left.\begin{array}{l} \frac{5a}{bc} \times b \times cd = 5abcd, \\ \frac{3c}{bc} = \frac{3}{b} \times bc \times cd = 3bccd, \\ \frac{4b}{cd} \times bc \times b = 4bbbc, \end{array}\right\} \text{the numerators.}$$

$$cd \times bc \times b = bbccd, \quad \text{the denominator.}$$

Therefore, $\frac{5a}{bc} = \frac{5abcd}{bbccd} = \frac{5ad}{bcd}$;

$$\frac{3c}{bc} = \frac{3bccd}{bbccd} = \frac{3cd}{bcd};$$

$$\frac{4b}{cd} = \frac{4bbbc}{bbccd} = \frac{4bb}{bcd}.$$

We obtain the values of the fractions given in the last column, by dividing all the terms by $b\,c$. *The fractions are thus reduced to their* LEAST *common denominator.*

Ans. $\frac{5ad+3cd+4bb}{bcd}$.

12. What is the sum of $\frac{3a}{2b}$, $\frac{4a}{3d}$ and $\frac{5a}{4bc}$?

13. Add together $\frac{4a}{b}$, $\frac{5b}{c}$ and $\frac{6c}{a}$.

14. Add $\frac{a+b}{5}$ and $\frac{2a+6}{8}$ together.

15. Add $a + \frac{x+y}{3}$ to $5a + \frac{3x+y}{7}$.

16. What is the sum of $6x$, $\frac{7x}{a}$, 5, and $\frac{3x+2}{4}$?

17. Add $\frac{4x}{a}$, $\frac{7x+1}{2}$, $\frac{3x+5}{b}$ and $2x + y$ together

18. Add $x - \frac{3a-2}{y}$, $3x - \frac{4a+7}{b}$, and $b + \frac{z-b}{x}$ together.

19. Add together ab, 15, $\frac{x+y}{c}$ and $\frac{6}{7}$.

20. Add together $\frac{a}{b}$, $\frac{c}{d}$, $4x$ and $\frac{5}{6}$.

21. Add together $\frac{1}{2}x$, and $\frac{3}{4}x$. ANS. $\frac{5x}{4}$.

It is evident that $\frac{1}{2}x$ expresses the same value as $\frac{x}{2}$, and that $\frac{3}{4}x$ is equivalent to $\frac{3x}{4}$; and $\frac{x}{2} + \frac{3x}{4} = \frac{5x}{4}$, or $1\frac{1}{4}x$.

22. Add together $\frac{1}{2}x$, and $\frac{2}{3}x$, and $\frac{3}{4}x$, and $\frac{5}{6}x$.

23. What is the sum of $\frac{1}{4}a$, $\frac{2}{3}a$, and $\frac{1}{2}a$?

24. Add together $\frac{2}{3}ab$, $\frac{5}{6}ab$, $\frac{3}{8}ab$, $\frac{1}{4}ab$, and $\frac{5}{9}abc$.

25. Add together $\frac{2}{3}(a-x)$, $\frac{3}{4}(a-x)$, and $\frac{1}{2}(a-x)$.

SECTION V.

Subtraction.

Subtraction is performed like addition; but *the signs of the quantities to be subtracted must all be changed.*

1. Subtract $\frac{3xy}{a}$ from $\frac{5xy}{a}$. Ans. $\frac{2xy}{a}$

2. Subtract $\frac{x}{y}$ from $\frac{z}{y}$.

3. Subtract $\frac{a}{b}$ from $\frac{c}{d}$.

4. Subtract $\frac{4am}{x}$ from $\frac{3bc}{5x}$.

5. Subtract $x+\frac{y-7}{4}$ from $3y+\frac{a-b}{c}$.

6. Subtract $\frac{3a+5}{8}$ from $\frac{7a}{2}$.

7. Subtract $\frac{4x+3}{x-1}$ from $\frac{6x-2}{x+1}$.

8. Subtract $\frac{1}{a+b}$ from $\frac{1}{a-b}$.

9. Subtract $\frac{x+y}{a}$ from $x-\frac{3-y}{b}$.

10. Subtract $6a-\frac{a+b}{7}$ from $9a-\frac{a-6}{4}$

11. Subtract $3y$ from $x-\frac{a+b}{4}$.

12. Subtract $4y$ from $5y-\frac{x-8xy}{x}$.

13. Subtract $\frac{2x-x}{4}$ from $\frac{8x+y}{x}$.

14. From $\frac{6}{a+m}$ take $\frac{a-m}{6}$.

15. Take $\frac{4x}{a}$ from $\frac{x}{5}$.

16. From $5ax$ take $\frac{3x}{7}$.

17. From $a+b$ take $\frac{a-b}{x}$.

18. Take $a-\frac{4+a}{3}$ from $x-y$

19. Subtract $a-b$ from $\frac{27x}{a+b}$.

SECTION VI.

Multiplication.

1. If a man gives to each of 3 beggars $\frac{1}{4}$ of a dollar, what does he give to the whole?

Ans. $\frac{3}{4}$ of a dollar.

For $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$, or $\frac{1}{4}\times 3=\frac{3}{4}$. If we substitute the algebraic fraction $\frac{a}{b}$ for $\frac{1}{4}$, and let c represent the number of beggars, the process will be the same: thus, $\frac{a}{b}\times c=\frac{ac}{b}$. That is, *we multiply the numerator by the integral quantity.*

2. Multiply $\frac{3a}{2bc}$ by $4x$. Ans. $\frac{6ax}{bc}$

3. Multiply $\frac{a+b}{7x}$ by $4xy$.

4. Multiply $\frac{a-2}{b+c}$ by $x-2$.

5. Multiply $\frac{4ab-6}{5x}$ by $5xy$.

6. Multiply $\frac{8ax-4b}{5y}$ by $-5ay$.

7. Multiply $\frac{a}{2b}$ by b. Ans. $\frac{a}{2}$.

In this example, instead of multiplying the numerator, which would give $\frac{b}{2b}$, we divide the denominator. In general, to multiply a fraction by an integral quantity, *we divide the denominator, when we can do so without a remainder; but when there would be a remainder, we multiply the numerator.*

When a fraction is multiplied by a quantity equal to its denominator, the numerator is the product; for, in that case, it is both divided and multiplied by the same quantity. Thus, $\frac{1}{2} \times 2 = 1$; $\frac{3}{4} \times 4 = 3$; $\frac{a}{b} \times b = a$; $\frac{x-z}{a+3} \times \overline{a+3} = x - z$, &c.

8. Multiply $\frac{3}{8}$ by 4. Ans. $\frac{3}{2} = 1\frac{1}{2}$.

9. Multiply $\frac{xy}{4bx}$ by $2b$. Ans. $\frac{xy}{2x} = \frac{y}{2}$

10. Multiply $\frac{3a-b}{21x+14y}$ by 7.

11. Multiply $\frac{8+a}{15b-10bx}$ by $5b$.

12. Multiply $\frac{a-x}{xy-2x}$ by $-x$.

13. Multiply $\frac{a}{b}$ by $\frac{c}{d}$. Ans. $\frac{ac}{bd}$

If we multiply $\frac{a}{b}$ by $\frac{c}{1}$, that is, by c, we obtain the product $\frac{ac}{b}$. But, as we are required to multiply by the quotient of c divided by d, and not by the whole value of c, this product is evidently too large, and must be divided by d. To multiply a fraction by a fraction, therefore, *we multiply the numerators for a new numerator, and the denominators for a new denominator.*

The value of a *Compound Fraction, which is the fraction of a fraction, as* $\frac{1}{2}$ *of* $\frac{3}{4}$, is found by this rule. Thus, $\frac{1}{2}$ of $\frac{3}{4}$ is $\frac{3}{8}$; and $\frac{1}{3}$ of $\frac{4}{5}$ is $\frac{4}{15}$.

14. Multiply $\frac{2c}{ab}$ by $\frac{3a}{2x}$.

15. Multiply $\frac{a+x}{y}$ by $\frac{3x}{4}$.

16. Multiply $\frac{a-b}{y}$ by $\frac{x}{b-4}$.

17. Multiply $\frac{a+b}{3+z}$ by $\frac{y-1}{x+b}$.

18. Multiply $\frac{ah-3}{4}$ by $\frac{bc}{a-x}$.

19. Multiply $\frac{axy}{b-8}$ by $\frac{b-8}{axy}$.

20. Multiply $\frac{3am-x}{2}$ by $\frac{3a}{x-y}$.

SECTION VII.

Division.

1. A man distributed $\frac{3}{4}$ of a dollar equally among 3 beggars. What did he give to each?

Ans. $\frac{1}{4}$ of a dollar.

One third part of three apples, three dollars, three units, or three fourths, is evidently one apple, one dollar, one unit, or one fourth; that is, $\frac{3 \div 3}{4} = \frac{1}{4}$. So, too, if $\frac{ac}{b}$ dollars be equally distributed among c persons, each one will have $\frac{ac \div c}{b} = \frac{a}{b}$ dollars. Here, to obtain the quotient, we divide the numerator by the integer.

2. Divide $\frac{3ab}{x}$ by $3a$. Ans. $\frac{b}{x}$.

3. Divide $\frac{x+y}{bc}$ by $x+y$.

4. Divide $\frac{8xy}{b-c}$ by $4y$.

5. Divide $\frac{14a}{3xy}$ by 7.

6. Divide $\frac{16a-8a}{a+2x}$ by $4a$.

7. A man divided $\frac{1}{2}$ a dollar equally between 2 persons. What was the share of each?

Ans. $\frac{1}{4}$ of a dollar.

Each person has $\frac{1}{2}$ of the sum given; and $\frac{1}{2}$ of $\frac{1}{2}$ is $\frac{1}{4}$. Here, the denominator is multiplied by the integer. In general, to divide a fraction by an integral quantity, *we divide the numerator, when we can do so without a remainder; and when we cannot, we multiply the denominator.*

8. Divide $\frac{ab}{c}$ by b. Ans. $\frac{a}{c}$

9. Divide $\frac{3xy}{b-c}$ by $3x$.

10. Divide $\frac{32x-8bx}{a+8}$ by $8x$

11. Divide $\frac{7ax}{y-z}$ by $3b$.

12. Divide $\frac{2m-x}{z}$ by $3y$.

13. Divide $\frac{8b-32xy}{x+8}$ by 4.

14. Divide $\frac{9}{10}$ by $\frac{3}{10}$. Ans. 3

As these fractions have the same denominator, we divide the numerator of the dividend by the numerator of the divisor. It is clear that $\frac{9}{10}$ contains $\frac{3}{10}$ 3 times.

15. Divide $\frac{a}{b}$ by $\frac{c}{d}$. Ans. $\frac{ad}{bc}$.

We first reduce the fractions to a common denominator, and we find that $\frac{a}{b} = \frac{a\,d}{b\,d}$, and $\frac{c}{d} = \frac{b\,c}{b\,d}$; and we then divide the numerator of the one by the numerator of the other, which gives us $\frac{a\,d}{b\,c}$. But if we multiply the numerator of the dividend by the denominator of the divisor, and the denominator of the dividend by the numerator of the divisor, we shall obtain the same result: thus, $\frac{a}{b} \times \frac{d}{c} = \frac{a\,d}{b\,c}$.

Hence, to *divide one fraction by another, we invert the divisor, and proceed as in multiplication.*

16. Divide $\frac{2a}{3x}$ by $\frac{b}{3y}$. Ans. $\frac{2ay}{bx}$.

17. Divide $\frac{3a-b}{c}$ by $\frac{b}{x-y}$.

18. Divide $\frac{8(a-b)}{x\,y}$ by $\frac{z(a+b)}{x\,y}$.

19. Divide $\frac{4ab}{x(a+8)}$ by $\frac{y\,z}{8(8m-n)}$

20. Divide $\frac{a+b}{x-y}$ by $\frac{8-a}{y+4}$.

21. Divide $\frac{a\,z-a}{x+y}$ by $a\,z$.

22. Divide $x\,y$ by $\frac{4x^2y-16x}{8a-12}$

23. Divide $\frac{-x}{y}$ by $\frac{a}{-b}$.

24. Divide $\frac{-x}{y}$ by $\frac{-a}{b}$.

25. Divide $\frac{3a}{b}$ by $\frac{x}{3}$.

26. Divide $\frac{x-y}{a}$ by $\frac{a-b}{x}$

CHAPTER VII.

POWERS.

SECTION I.

Involution of Simple Quantities.

1. WHAT is the product of $a\,a$ multiplied by a?

ANS. a^3.

The answer to this question, obtained by the common process of multiplication, is $a\,a\,a$. Instead of repeating the letter, we indicate the number of times it occurs in the product, by the small figure placed above it on the right hand. *This figure is called its Exponent.* It shows the *Power* of the letter; that is, *how many times it is used as a factor in the multiplication.* When a letter has no number annexed, the exponent is always a unit, or 1, and the letter is said to be the first power or *Root;* thus,

a is the root, or first power;
$a \times a = a^2$ is the second power;
$a \times a \times a = a^3$ is the third power;
$a \times a \times a \times a = a^4$ is the fourth power, &c.

The second power is sometimes called the square; the third power, the cube; and the fourth power, the

biquadrate. If we suppose the value of a to be 3,

$$a = 3, \text{ the first power;}$$
$$a^2 = 3^2, \text{ or } 3 \times 3 = 9, \text{ the second power;}$$
$$a^3 = 3^3, \text{ or } 3 \times 3 \times 3 = 27, \text{ the third power;}$$
$$a^4 = 3^4, \text{ or } 3 \times 3 \times 3 \times 3 = 81, \text{ the fourth power, \&c.}$$

The *coefficient* and the *exponent* of a quantity, being very different things, must not be confounded together. Thus, the values of $5\,x$ and x^5 are far from being equal. Let the value of x be 6;

then $5\,x = 5 \times 6 = 30$;

but $x^5 = 6 \times 6 \times 6 \times 6 \times 6 = 7776$.

It is evident that *we raise a single letter to any proposed power by giving it the exponent of that power.*

2. What is the sixth power of x? the fourth power of d? the fifth power of c? the eighth power of a? the seventh power of x?

3. What is the product of $a\,b$ multiplied by $a\,b$; that is, what is the second power of $a\,b$?

$$a\,b \times a\,b = a\,a\,b\,b, \text{ or } a^2\,b^2.$$

Here, the quantity to be raised consists of two factors, a and b; and *the required power is expressed by the same factors, with the exponent of that power written above each.*

4. What is the fourth power of $x\,y$?

5. Raise $a\,b\,c$ to the sixth power.

6. What is the ninth power of $a\,b\,x$?

7. Involve $m\,n\,y$ to the seventh power.

8. What is the square of $4\,a\,b$? Ans. $16\,a^2\,b^2$.

It will be remembered, that *the square, or second power of any quantity, is the product of that quantity*

multiplied by itself; consequently, the second power of $4\,a\,b$ is $4\,a\,b \times 4\,a\,b = 16\,a\,a\,b\,b$, or $16\,a^2\,b^2$.

Hence, *coefficients must be raised to any required power, by actual multiplication.*

9. What is the third power of $5\,a\,x$?
10. What is the fourth power of $7\,a\,b\,c$?
11. Raise $6\,x\,y\,z$ to the fifth power.
12. What is the fourth power of $-2\,a\,b\,c$?

$$\begin{array}{l} -2\,a\,b\,c, \text{ first power.} \\ \underline{-2\,a\,b\,c} \\ +4\,a^2\,b^2\,c^2, \text{ second power.} \\ \underline{-2\,a\,b\,c} \\ -8\,a^3\,b^3\,c^3, \text{ third power} \\ \underline{-2\,a\,b\,c} \\ +16\,a^4\,b^4\,c^4, \text{ fourth power.} \end{array}$$

Hence it appears, that *when the root or first power is a negative quantity, the* ODD *powers are negative, and the* EVEN *powers are positive.*

13. What is the second power of a^3? ANS. a^6.

Here, the quantity to be involved is already a power, and *the exponent is multiplied by the exponent of the power proposed*; thus, $a^{3\times 2} = a^6$; for $a^3 = a\,a\,a$, and $a\,a\,a \times a\,a\,a = a^6$.

14. What is the third power of $a\,b^3$? ANS. $a^3\,b^9$
15. What is the second power of $a^3\,b\,x^2$?
16. Raise $a^2\,b^3\,x$ to the third power.
17. What is the fourth power of $m^4\,x^2\,y^5$?
18. Involve $6\,a^2\,c^3\,x^4$ to the second power.
19. What is the third power of $3\,x^2\,y^3\,z^4$?
20. Required the fourth power of $4\,a^2\,x^3\,y^2\,z$.

SECTION II.

Involution of Compound Quantities.

1. It is required to find the fifth power of the binomial quantity $a + b$.

$a + b$, first power.
$a + b$

$a^2 + ab$
$\quad\quad ab + b^2$

$a^2 + 2ab + b^2$, second power.
$a + b$

$a^3 + 2a^2b + ab^2$
$\quad\quad a^2b + 2ab^2 + b^3$

$a^3 + 3a^2b + 3ab^2 + b^3$, third power.
$a + b$

$a^4 + 3a^3b + 3a^2b^2 + ab^3$
$\quad\quad a^3b + 3a^2b^2 + 3ab^3 + b^4$

$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, fourth power.
$a + b$

$a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4$
$\quad\quad a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5$

$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, Ans.

The powers of compound quantities are obtained by actual multiplication. These powers, however, are often expressed by means of an exponent. Thus, $(a + b + c)^3$ indicates the third power of $a + b + c$

2. It is required to find the fifth power of the residual quantity $a - b$.

$a - b$, first power.
$a - b$

$a^2 - ab$
$\quad - ab + b^2$

$a^2 - 2ab + b^2$, second power.
$a - b$

$a^3 - 2a^2b + ab^2$
$\quad - a^2b + 2ab^2 - b^3$

$a^3 - 3a^2b + 3ab^2 - b^3$, third power.
$a - b$

$a^4 - 3a^3b + 3a^2b^2 - ab^3$
$\quad - a^3b + 3a^2b^2 - 3ab^3 + b^4$

$a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$, fourth power
$a - b$

$a^5 - 4a^4b + 6a^3b^2 - 4a^2b^3 + ab^4$
$\quad - a^4b + 4a^3b^2 - 6a^2b^3 + 4ab^4 - b^5$

$a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$, Ans

3. What is the second power of $3a - b + 2c$?

$3a - b + 2c$
$3a - b + 2c$

$9a^2 - 3ab + 6ac$
$\quad - 3ab + b^2 - 2bc$
$\quad + 6ac - 2bc + 4c^2$

$9a^2 - 6ab + 12ac + b^2 - 4bc + 4c^2$.

4. What is the third power of $a+1$?
5. Raise $x+y+z$ to the second power.
6. Involve $a+b+c+d$ to the second power.
7. What is the third power of $2a-x+c^2$?
8. Involve $3a+2b$ to the third power.
9. What is the second power of $a+b-c$?
10. What is the third power of $a-x+8$?
11. What is the second power of $\frac{a}{b}$? Ans. $\frac{a^2}{b^2}$.

For $\frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb}$, or $\frac{a^2}{b^2}$, according to the principles already explained. Therefore, *we involve a fraction, by raising both the numerator and denominator to the required power.*

12. What is the second power of $\frac{a^2b}{x^2y}$? Ans. $\frac{a^4b^2}{x^4y^2}$
13. What is the second power of $\frac{3a^2b^3}{xy^2}$?
14. What is the second power of $\frac{4a^3b^2x}{3d^2y}$?
15. What is the second power of $\frac{a+b}{x-y}$?
16. What is the third power of $\frac{a-5}{x}$?
17. Raise $\frac{3x^2y}{2a^3}$ to the third power.
18. Required the second power of $\frac{2a-3}{b^2+c}$.
19. Required the fourth power of $\frac{4}{2x^2y}$.
20. Required the second power of $\frac{5x^2-a^2}{9}$.
21. What is the second power of $\frac{y-x}{a^2b}$?

SECTION III.

The Binomial Theorem.

The first and second examples of the last section exhibit the method of involving binomial and residual quantities, by actual multiplication. It is very apparent that, when a high power is required, the process must become tedious. From these operations, however, may be derived rules for raising such quantities to any proposed power, without the intervention of the lower powers. This method of involving quantities was discovered by Sir Isaac Newton, and is called the *Binomial Theorem.*

When we would express any required power of a given binomial or residual quantity, four things require attention; namely, *the number of terms in that power, the signs, the exponents, and the coefficients.*

THE TERMS. It will be observed, that *the number of terms in every power, is greater by one than the exponent of that power.* Thus, the second power consists of three terms; the third power, of four terms; the fourth power, of five terms, &c.

THE SIGNS. The corresponding powers of these two quantities, $a+b$ and $a-b$, differ only in their signs. *All the terms of the binomial quantity are positive; whereas the* EVEN *terms of the residual quantity are negative, and the* ODD *terms positive.*

THE EXPONENTS. *The first term of every power consists of the first letter of the given binomial, a,*

raised to that power; and the other exponents of that letter diminish by one towards the right. The first letter of the binomial, a, is called the *leading* quantity; and its exponents for the seventh power, are

7, 6, 5, 4, 3, 2, 1.

The last term of every power is the last letter of the given binomial, b, raised to that power; and the other exponents of that letter diminish by unity towards the left. The last letter of the binomial, b, is called the *following* quantity; and its exponents for the seventh power, are

1, 2, 3, 4, 5, 6, 7.

The sum of the exponents of the two letters, is always equal to the exponent of the power in which they occur.

To apply these principles, let it be required to determine the number of terms, the signs and exponents of $a+b$ and $a-b$, in the seventh power. The number of terms will be eight, that is, one more than the power proposed; and their signs and exponents, without the coefficients, will stand thus:

$$(a+b)^7=a^7+a^6b+a^5b^2+a^4b^3+a^3b^4+a^2b^5+ab^6+b^7$$
$$(a-b)^7=a^7-a^6b+a^5b^2-a^4b^3+a^3b^4-a^2b^5+ab^6-b^7$$

THE COEFFICIENTS. It will be seen, by an inspection of the examples above referred to, *that the coefficient of the first term is always* 1, *and that the coefficient of the second term is the exponent of the power.* The remaining coefficients may be found by the following RULE: *If the coefficient of any term be multiplied by the exponent of the leading quantity in that*

term, and the product be divided by the number which denotes its place from the left, it will give the coefficient of the next term.

The coefficients of $a + b$, raised to the seventh power, are

$$1,\ 7,\ \frac{7\times 6}{2},\ \frac{21\times 5}{3},\ \frac{35\times 4}{4},\ \frac{35\times 3}{5},\ \frac{21\times 2}{6},\ \frac{7\times 1}{7}.$$

$$1,\ 7,\ 21,\ 35,\ 35,\ 21,\ 7,\ 1.$$

If we prefix these coefficients to the terms of $a + b$ and $a - b$, already obtained, we shall have the seventh powers of those quantities complete:

$$(a + b)^7 = a^7 + 7\,a^6\,b + 21\,a^5\,b^2 + 35\,a^4\,b^3 + 35\,a^3\,b^4 + 21\,a^2\,b^5 + 7\,a\,b^6 + b^7.$$

$$(a - b)^7 = a^7 - 7\,a^6\,b + 21\,a^5\,b^2 - 35\,a^4\,b^3 + 35\,a^3\,b^4 - 21\,a^2\,b^5 + 7\,a\,b^6 - b^7.$$

It will be observed, that the coefficients are equal in the first and last terms, also in the second and last but one, the third and last but two, and so on. It will be sufficient, therefore, in practice, to find the coefficients of half the terms, if their number be even, or of one more than half, if it be odd, and apply them to the rest.

3. What is the fourth power of $x + y$?

   ANS. $x^4 + 4\,x^3\,y + 6\,x^2\,y^2 + 4\,x\,y^3 + y^4$

4. Raise $x - y$ to the sixth power.
5. What is the fifth power of $a + c$?
6. What is the eighth power of $m - n$?
7. What is the fourth power of $m + n$?
8. Required the seventh power of $x + y$.
9. What is the ninth power of $a + b$?
10. Raise $c + d$ to the tenth power.

11. Required the thirteenth power of $a - b$.
12. Required the sixth power of $x + z$.
13. What is the seventh power of $c - d$?

14. What is the third power of $a + 3\,b\,c$?

This quantity, in its present form, can be involved only by actual multiplication; for the rules given above require each term of the binomial to consist of a single letter, whose coefficient must be 1. But if the value of the second term, $3\,b\,c$, be represented by some letter, a binomial will be formed, which can be involved in the usual manner.

Suppose m, for instance, to be equivalent to $3\,b\,c$; then, instead of $a + 3\,b\,c$, we have the binomial $a + m$, the third power of which is

$$a^3 + 3\,a^2\,m + 3\,a\,m^2 + m^3.$$

In the place of m, put down its value, and the required power will be obtained. We supposed $m = 3\,b\,c$; therefore, $m^2 = 9\,b^2\,c^2$, and $m^3 = 27\,b^3\,c^3$.

The first term is a^3.
The second term is $3\,a^2\,m$, or $3\,a^2 \times 3\,b\,c = 9\,a^2\,b\,c$.
The third term is $3\,a\,m^2$, or $3\,a \times 9\,b^2\,c^2 = 27\,a\,b^2\,c^2$.
The fourth term is $m^3 = 27\,b^3\,c^3$.

ANS. $a^3 + 9\,a^2\,b\,c + 27\,a\,b^2\,c^2 + 27\,b^3\,c^3$.

15. What is the fourth power of $2\,a\,b - x$?

Suppose $m = 2\,a\,b$; and involve $m - x$. The fourth power of this binomial is

$$m^4 - 4\,m^3\,x + 6\,m^2\,x^2 - 4\,m\,x^3 + x^4.$$

In the place of m, use its value; observing that $m = 2\,a\,b$, $m^2 = 4\,a^2\,b^2$, $m^3 = 8\,a^3\,b^3$, and $m^4 = 16\,a^4\,b^4$.

ANS. $16\,a^4\,b^4 - 32\,a^3\,b^3\,x + 24\,a^2\,b^2\,x^2 - 8\,a\,b\,x^3 + x^4$

16. What is the second power of $4\,a\,b - 5\,c^2$?

Let $m = 4\,a\,b$, and $n = 5\,c^2$; and, instead of the given binomial, involve $m + n$.

The second power of $m + n$ is

$$m^2 + 2\,m\,n + n^2.$$

Instead of m and n, put their respective values into this power. Observe, $m = 4\,a\,b$, and $m^2 = 16\,a^2\,b^2$; also $n = 5\,c^2$, and $n^2 = 25\,c^4$.

Ans. $16\,a^2\,b^2 + 40\,a\,b\,c^2 + 25\,c^4$.

17. What is the third power of $x + y + z$?

Let $m = y + z$, and involve $x + m$, which is equivalent to $x + y + z$.

The third power of $x + m$ is

$$x^3 + 3\,x^2\,m + 3\,x\,m^2 + m^3.$$

To obtain the required power, we must restore the value of m.

$$m = y + z.$$
$$m^2 = y^2 + 2\,y\,z + z^2.$$
$$m^3 = y^3 + 3\,y^2\,z + 3\,y\,z^2 + z^3.$$

Ans. $x^3 + 3\,x^2\,y + 3\,x^2\,z + 3\,x\,y^2 + 6\,x\,y\,z + 3\,x\,z^2 + y^3 + 3\,y^2\,z + 3\,y\,z^2 + z^3$.

18. What is the second power of $a + b + c + d$?

Let $m = a + b$, and $n = c + d$; and then involve $m + n$.

$$(m + n)^2 = m^2 + 2\,m\,n + n^2.$$

In this quantity, substitute the values of m and n respectively, and the required power will be complete.

$$m^2 = a^2 + 2\,a\,b + b^2.$$
$$n^2 = c^2 + 2\,c\,d + d^2.$$

Ans. $a^2 + 2\,a\,b + b^2 + 2\,a\,c + 2\,b\,c + 2\,a\,d + 2\,b\,d + c^2 + 2\,c\,d + d^2$.

In this way, any quantity whatever can be raised to any required power, by means of the binomial theorem. *The given quantity must first be reduced to a binomial, consisting of two single letters; and after it has been involved, the respective values of these letters must be restored.*

19. What is the second power of $a - 3\,b$?
20. Required the third power of $5\,a^2 + b$.
21. What is the second power of $3\,a^2 + 5\,b^3$?
22. What is the second power of $a + b^2 - 7$?
23. Required the third power of $2\,x - y + z^2$.
24. What is the fifth power of $a^2 - c - 2\,d$?
25. What is the fifth power of $7\,x^2\,y^5 - 10\,x^5\,z^2$?

SECTION IV.

Addition.

1. What is the sum of a^2 and b^2? ANS. $a^2 + b^2$

As the quantities a and b are dissimilar, any powers of these quantities must also be dissimilar; and they can only be added by means of the sign $+$.

2. What is the sum of a^2 and a^3? ANS. $a^2 + a^3$.

It is evident that different powers of the same letter are dissimilar quantities, and must be added as above. If we suppose the value of a to be 4, $a^2 = 4 \times 4$, or 16; and $a^3 = 4 \times 4 \times 4 = 64$; and $a^2 + a^3 = 16 + 64$, or 80. But $2\,a^2 = 16 \times 2$, or 32; and $2\,a^3 = 64 \times 2$, or 128; therefore, neither $2\,a^2$ nor $2\,a^3$ is the true sum of a^2 and a^3.

3. What is the sum of $2\ a^2$ and $3\ a^2$? Ans. $5\ a^2$.

For $2\ a^2 = 2 \times a^2$, and $3\ a^2 = 3 \times a^2$; and $(2+3) \times a^2 = 5\ a^2$. Let the value of a be 4; then $2\ a^2 = 2 \times 4 \times 4$, or 32; and $3\ a^2 = 3 \times 4 \times 4$, or 48; and $32 + 48 = 80$; but $5\ a^2 = 5 \times 4 \times 4$, or 80 also.

Observe that the exponents are not changed, when the powers of quantities are added together.

From all which we infer, that *the same letters under the same exponents are similar quantities, and must be added by the common rules; but different letters, or the same letters under different exponents, are dissimilar quantities, and must be added by means of the sign* $+$.

4. What is the sum of a^3 and b^3? Ans. $a^3 + b^3$.
5. What is the sum of a^3 and a^4?
6. Add together $2\ a^2$, $3\ a^2$, $4\ a^2$, and $5\ a^2$.
7. What is the sum of $2\ a^3$, x^2, $3\ a^3$, and $5\ x^2$?
8. What is the sum of $a\ x^2$, $b\ x^2$, $c\ x^2$, and $d\ x^2$?

9.) $$\begin{array}{r} 5\,a^3 b + 3\,a^2 b c - 7\,a b^3 \\ -8\,a^3 b - 8\,a^2 b c + 7\,a b^3 \\ 9\,a^3 b + 5\,a^2 b c - \ \ a b^3 \\ \hline \end{array}$$

10.) $$\begin{array}{r} a^3 x^2 + b\,y^2 \\ c d\ x^2 + d\,y^2 \\ -2\,a^3 x^2 - d\,y^2 \\ \hline \end{array}$$

11. What is the sum of $a\ (x^2 - y^3)^2$, $3\ a\ (x^2 - y^3)^2$, and $5\ a\ (x^2 - y^3)^2$?

12. What is the sum of $8\,(a^2 - b^3)^2$ and $4\ (a^2 - b^3)^2$?

13. Add together $5\ a^2\ b\ c^3$, $3\ a^2\ b\ c^3$, $a^2\ b\ c^3$, and $2\ a^2\ b^3\ c$.

14. Required the sum of $3\ a^3 + b\ c^2$, $5\ a^3 + 2\ b\ c^2$, $a^3 + 5\ b\ c^2$, and $6\ a^3 + 2\ b\ c^2$.

15 What is the sum of $\frac{1}{2}\ (a^2 - b\ x)^3$, $\frac{3}{4}\ (a^2 - b\ x)^3$, $3\ (a^2 - b\ x)^3$, and $\frac{1}{4}\ (a^2 - b\ x)^3$?

16 Add together $a^2 b^3 + 7 + x^2$, $3 a^2 b^3 + 9$, $x^2 + 14$, $3 a^2 b^3 - 19 - 5 x^2$, $12 - 2 a^2 b^3$, and $4 a^2 b^3 - 8 + 2 x^2$.

17. Required the sum of $a^3 (b^2 - 12 a^3 + y^4)^3$, $3 a^3 (b^2 - 12 a^3 + y^4)^3$, $2 a^3 (b^2 - 12 a^3 + y^4)^3$, and $8 a^3 (b^2 - 12 a^3 + y^4)^3$.

18. What is the sum of $x^2 y^3 + 14 - 13 b^2$, and $12 + 3 x^2 y^3 - 8 b^2$, and $15 b^2 - 20 - x^2 y^3$?

SECTION V.

Subtraction.

1. What is the difference between a^3 and b^2?

Ans. $a^3 - b^2$.

The questions in this section are solved like those in the last. The signs of the quantities to be subtracted must, however, be changed.

Observe, that the exponents remain unaltered in subtraction, as in addition.

2. Subtract $2 a^3$ from $5 a^3$. Ans. $3 a^3$.
3. From $5 (a^2 - b^3)$ take $3 (a^2 - b^3)$.
4. From $6 a^2 (a^3 - b^2 + 3)^2$ take $5 a^2 (a^3 - b^2 + 3)^2$
5. Subtract $3 a^5 b^3$ from $7 a^5 b^3$.
6. Subtract $2 a^2$ from $3 a^3$.
7. Subtract $3 a^3$ from $5 a^3$.
8. Subtract $5 a^2 b$ from $2 a^2 x$.
9. Subtract $7 a^2 x$ from $9 a^2 y$.
10. Subtract $5 a^2 x + 3 d^2 x$ from $8 a^2 x - 4 d^2 x$.
11. From $8 a^2 x y - x^3$ take $5 a^2 x y + 2 x^3$

12. From $9a^2x^3 - 12 + 20a^2bx^3$
Take $\underline{3a^2x^3 + 16 - 12a^2bx^3.}$

13.) $17a^2x^3 + 5xy^2$
$\underline{12a^2x^3 - 4xy^2}$

14. From $5(a^2 - b^3)^2 + x^2y$
Take $\underline{3(a^2 - b^3)^2 + 2x^2y.}$

15.) $2x(a-b)^3 + (a-b)^3$
$\underline{x(a-b)^3 + (a-b)^3}$

16. From $3a^3(b^2 - 8)^3$ take $a^3(b^2 - 8)^3$.

17. Subtract $2x^2(b^2 - c^3)$ from $5x^2(b^2 - c^3)$.

18. Take $3a^2b - 3x^2y^3 + 8$ from $8a^2b + x^2y^3 - 17$.

19. From $5x^3y(ab^2 - 17)^3$ take $3x^3y(ab^2 - 17)^3$.

20. Subtract $6ab^2c^3 - b^2c^3 + 7a$ from $8ab^2c^3 + 2b^2c^3 - 3a^2$.

SECTION VI.

Multiplication.

1. What is the product of $4a^2$ multiplied by $5a^3$?

ANS. $20a^5$.

This product may be expressed thus, $20a^3a^2$; that is, $20aaaaa$, or $20a^5$. But as the sum of the exponents is 5, the exponent of the product may be obtained, at once, by adding together the exponents of the factors. Thus, $a^2 \times a^3 = a^{2+3}$, or a^5; also $x^4 \times x^5 = x^{4+5}$, or x^9; and $a^2b^3 \times a^3b^4 = a^{2+3}b^{3+4}$, or a^5b^7.

Hence, when the same letter, under any power whatever, occurs in both the multiplier and multipli

cand, *add together its exponents, and the sum will be the exponent of the product.*

2. Multiply $3\ a^2\ b$ by $6\ a\ b^3$. ANS. $18\ a^3\ b^4$.
3. Multiply $8\ a^3\ b^2\ x$ by $2\ a^2\ b^2\ x$.
4. Multiply $3\ a^5\ b^2\ x\ y$ by $5\ a\ b^3\ x^3\ y^5$.
5. Multiply $9\ a^3\ c^2\ m$ by $9\ a^3\ c^2\ m$.
6. Multiply $x^3 + 5\ a$ by x^2.
7. Multiply $3\ a^2 - 5\ a\ x^3$ by $2\ a^3\ x^2$.
8. What is the product of $a\ y - 6 + y^3$ multiplied by $a^3\ y^2$?
9. Required the product of $a^3 - b^2$ multiplied by $a^3 + b^2$.
10. Multiply $a^2\ b\ c^3 - 12$ by $2\ a^2\ b^3$.
11. Multiply $x^3 + x^2\ y + x\ y^2 + y^3$ by $x - y$.
12. Required the product of $a^3 + a - 6$ multiplied by $2\ a^2 + a + 1$.
13. Multiply $a^2 + a^4 + a^6$ by $a^2 - 1$.
14. Multiply $3\ (x^2 - y^3)^2$ by $a\ (x^2 - y^3)^2$.
ANS. $3\ a\ (x^2 - y^3)^4$.
15. Required the product of $a^2\ (x^3 - 12 + x^2)^3$ multiplied by $a\ b\ (x^3 - 12 + x^2)^3$.
16. Multiply $3\ (a^2 + b^3)^2$ by $5\ (a^2 + b^3)^2$.
17. Multiply $a\ (x^3 - y^2 + 4)^3$ by $6\ (x^3 - y^2 + 4)^3$
18. Multiply $3\ a^2\ (x^3 - y^3)^2$ by $4\ a^2\ (x^3 - y^3)^2$.
19. Multiply $4\ a^2 - 16\ a\ x + 3\ x^2$ by $5\ a^3 - 2\ a^2\ x$.
20. Multiply $a^4 - 2\ a^3\ b + 4\ a^2\ b^2 - 8\ a\ b^3 + 16\ b^4$ by $a + 2\ b$.
21. Multiply $2 + 6\ a^2 - b^3$ by $x^2 - a$.
22 Multiply $\frac{x^3 - y^2}{a^2 + b}$ by $\frac{x^3 + y^2}{a^2 - 8}$.

SECTION VII.

Division.

1. Divide a^5 by a^3 Ans. a^2.

The exponent of the quotient added to that of the divisor, must be equal to the exponent of the dividend; for these two quantities are multiplied together by the adding of their exponents; and the product of the divisor and quotient must always give the dividend.

In the above example, the dividend $a^5 = a\,a\,a\,a\,a$, and the divisor $a^3 = a\,a\,a$; and the division may be expressed thus, $\frac{a\,a\,a\,a\,a}{a\,a\,a}$. Now, if we cancel $a\,a\,a$ in the numerator and denominator, there will be left in the dividend $a\,a$, that is, a^2, as in the answer given above.

Hence, if we are required to divide a power of any letter, as a^5, by another power of the same letter, as a^3, *we must subtract the exponent of the divisor from the exponent of the dividend, and the remainder will be the exponent of the quotient.*

2. Divide a^6 by a^3. Ans. a^3.
3. Divide $a^3\,b^4$ by $a\,b^3$.
4. Divide $a^6\,b^5\,c^2$ by $a^6\,b^4\,c$.
5. Divide $16\,a^{10}\,b^4\,x^2$ by $4\,a^5\,b^3\,x^2$.
6. Divide $39\,a^2\,m^3\,y^4$ by $13\,a^2\,m^2\,y^3$.

7. Divide a^2 by a^2. Ans. 1

By the rule above obtained, $a^2 \div a^2 = a^0$; for

$a^{2-2} = a^0$. But $\frac{a^2}{a^2} = 1$; for if we divide any quantity by itself, the quotient is 1. Hence we see, that *a^0 is always equal to* 1, *whatever may be the value of a;* and the same is true of any other quantity which has ^{0} for its exponent.

8. Divide a^5 by a^5.

9. Divide 5 a^3 by 5 a^3.

10. Divide a^2 by a^3. Ans. a^{-1}.

Here the exponent of the divisor is greater than that of the dividend; but the general rule must be observed. Thus, $a^2 \div a^3 = a^{2-3} = a^{-1}$. In the expression a^{-1}, a is said to have a *negative exponent.* So, also, $a \div a^2 = a^{-1}$; $a \div a^3 = a^{-2}$; $a \div a^4 = a^{-3}$; $a \div a^5 = a^{-4}$; and so on.

11. Divide a^7 by a^9. Ans. a^{-2}

12. Divide $a^2\, b^4\, x^3$ by $a^5\, b^6\, x^7$.

13. Divide a^0 by a. Ans. a^{-1}.

The division of a^0 by a, that is, by a^1, may be thus expressed; $a^0 \div a^1 = a^{0-1} = a^{-1}$. But it will be remembered, that $a^0 = 1$; therefore, $\frac{a^0}{a}$ is the same as $\frac{1}{a}$; and, consequently, $a^{-1} = \frac{1}{a}$.

Hence we learn the value of those powers which have negative exponents. Thus, $a^0 = 1$; $a^{-1} = \frac{1}{a}$; $a^{-2} = \frac{1}{a^2}$; $a^{-3} = \frac{1}{a^3}$; $a^{-4} = \frac{1}{a^4}$; $a^{-5} = \frac{1}{a^5}$; &c.

If the value of a be 2, then $a^0 = 1$; $a^{-1} = \frac{1}{2}$; $a^{-2} = \frac{1}{4}$; $a^{-3} = \frac{1}{8}$; $a^{-4} = \frac{1}{16}$; $a^{-5} = \frac{1}{32}$; &c.

14. Divide $(a+b)^3$ by $(a+b)^3$. Ans. 1.

15 Divide $(a^2 - x^3 + b)^4$ by $(a^2 - x^3 + b)^2$

16. Divide $(a^3+5-y)^3$ by $(a^3+5-y)^7$.

17. Divide $a\,b^3+2\,a^3\,b+2\,a^2\,b^2+a^4$ by $a\,b^2+a^3+a^2\,b$.

Before we begin to divide compound quantities, we should arrange the terms of the divisor and dividend according to the powers of their letters, as this will greatly facilitate the work. The highest power of a letter should come first, and the lower powers should succeed in order. The first term of the divisor and the first term of the dividend should contain the same letter.

To arrange this question, we place the letter of the divisor which is of the highest power, first, and the other terms in order, thus; $a^3+a^2\,b+a\,b^2$. Now, as the first term of the divisor is a, the first term of the dividend should also contain a, and the whole should be arranged thus; $a^4+2\,a^3\,b+2\,a^2\,b^2+a\,b^3$.

$$
\begin{array}{r|l}
a^3+a^2\,b+a\,b^2\,) & a^4+2\,a^3\,b+2\,a^2\,b^2+a\,b^3\ (\,a+b. \\
 & \underline{a^4+\ \ a^3\,b+\ \ a^2\,b^2\qquad\quad} \\
 & \qquad\ \ a^3\,b+\ \ a^2\,b^2+a\,b^3 \\
 & \qquad\ \ \underline{a^3\,b+\ \ a^2\,b^2+a\,b^3} \\
 & \qquad\quad * \qquad\ * \qquad\ *
\end{array}
$$

18. Divide $a\,b^3-b^3\,c$ by $a-c$.

19. Divide $3\,a^5+16\,a^4\,b-33\,a^3\,b^2+14\,a^2\,b^3$ by $a^2+7\,a\,b$.

20. Divide $x^3+9\,x^2+4\,x-80$ by $x+5$.

21. Divide $b^8-16\,c^8$ by $b^2-2\,c^2$.

22. Divide $a^2\,x-b^2\,x+8\,x-a^2\,y^3+b^2\,y^3-8\,y^3$ by $x-y^3$.

23. Divide $a^6-a^4\,x-a^2\,x^3+2\,x^4$ by a^4-x^3.

CHAPTER VIII.

EQUATIONS OF THE FIRST DEGREE.

SECTION I.

Introduction.

WHEN *two equal quantities, differently expressed, are compared together by means of the sign = between them, such an expression is called an equation.* Thus, $8+4=18-6$ is an equation; for the sums are equal, though expressed in different numbers. So, too, if $x+5$ and $a-7$ represent equal quantities, we have the equation $x+5=a-7$.

It is by means of equations that most of the investigations of Algebra are carried on; and the preceding chapters may be regarded as merely preparatory to this part of the science.

An equation of the first degree contains only the first power of the unknown quantity, as x. When some higher power of the unknown quantity, as x^2, or x^3, enters into the equation, it is said to be of the second or third degree.

The terms on the left of the sign =, taken together, are called the first member of the equation; those on the right, the second member. Thus, in the equation

$x + 6 = a - 5$, the first member is $x + 6$, and $a - 5$ is the second.

Any changes, which convenience requires, may be made in the members of an equation, provided the same be made in both, so that their equality is preserved; as may be seen in the following examples:

Given the equation $8 + 4 = 18 - 6$.

Add 10 to each member;

$$8 + 4 + 10 = 18 - 6 + 10.$$

Subtract 12 from each member;

$$8 + 4 + 10 - 12 = 18 - 6 + 10 - 12.$$

Multiply each term by 2;

$$16 + 8 + 20 - 24 = 36 - 12 + 20 - 24.$$

Divide every term by 4;

$$4 + 2 + 5 - 6 = 9 - 3 + 5 - 6$$

Although the members of the given equation, $8 + 4 = 18 - 6$, are changed in form and value by each successive operation, it will be seen that their equality is preserved throughout. Whence we infer that,

The same quantity may be added to both members of an equation;

The same quantity may be subtracted from both members of an equation;

All the terms of an equation may be multiplied by the same quantity; and

All the terms of an equation may be divided by the same quantity; without affecting, in either of these cases, the equality of the two members.

The application and use of these and other changes in the terms and members of equations, will be ex-

plained hereafter, as it is found necessary to introduce them.

When a question is proposed, to be solved by Algebra, the first step is, to express its conditions in the form of an equation; or, in common language, to *put the question into an equation.*

The next step is, to find the value of the unknown quantity from the terms with which it is associated; which process is called *the reducing or resolving of an equation.*

The rules for reducing equations are few and simple; and they will be given in the subsequent sections. But no particular mode of putting questions into an equation, can be prescribed, as the process must vary with the conditions of every question. The following general directions may be of some service:

When a question is proposed, before its solution is attempted, get a clear and distinct understanding of its nature and design.

Let the thing required, the answer to the question, be represented by some letter, as x or y.

Regard the letter used as the answer of the question, and perform the same operations on it as would be necessary to *prove* the real answer to be correct.

The result thus obtained will be one member of the equation; and the other member will be found in the corresponding conditions of the question.

To illustrate these general directions by a single example: Let us suppose that a man gave 175 dollars for his watch, chain and seal; that the chain cost twice as much as the seal; and the watch twice as uch as the chain. What was the price of each?

This question has three required quantities, or answers, namely, the prices of the watch, the chain, and the seal, either of which may be assumed and represented by a letter, as x; and, either being known, the others can readily be found.

First, let us suppose the *watch* cost x dollars; then the chain cost $\frac{x}{2}$ dollars, or half as much as the watch; and the seal cost $\frac{x}{4}$ dollars, or half as much as the chain; and, according to the answer assumed, they all cost $x + \frac{x}{2} + \frac{x}{4}$ dollars. But they actually cost, by the question, 175 dollars; we have, therefore, this equation:

$$x + \frac{x}{2} + \frac{x}{4} = 175.$$

Again, let us suppose the *chain* cost x dollars; then the watch cost twice as much, or $2x$ dollars; and the seal cost half as much as the chain, or $\frac{x}{2}$ dollars. According to this supposition, the cost of the whole was $2x + x + \frac{x}{2}$ dollars; and we have this equation:

$$2x + x + \frac{x}{2} = 175.$$

Finally, let us suppose the *seal* cost x dollars; then the chain cost $2x$ dollars; and the watch cost twice as much as the chain, or $4x$ dollars. By this supposition, the price of the whole was $4x + 2x + x$ dollars; and we have the following equation:

$$4x + 2x + x = 175.$$

Either of these equations will give one answer to

the question, from which the other answers may be obtained by common multiplication or division. The last form, being the most simple, is to be preferred.

It will be observed, that the value of x is different in each of these equations. In the first, it is 100 dollars, the price of the watch; in the second, it is 50 dollars, the price of the chain; in the third, it is 25 dollars, the price of the seal.

To find the numerical value of the unknown quantity, we make it stand alone, neither multiplied nor divided by any number or quantity, as one member of the equation; and collect all the known quantities together for the other member.

SECTION II.

To remove Coefficients.

1. A man gave 381 dollars for a horse and chaise, and the chaise cost twice as much as the horse. Required the price of each.

Let x represent the price of the horse. Then $2x$ will be the price of the chaise; and $x + 2x$ will be the price of both. We have, therefore, the following equation:

$$x + 2x = 381.$$

Add the x's, $\quad 3x = 381.$

Divide by 3, $\quad x = 127$, the price of the horse.

And $\quad 2x = 254$, the price of the chaise.

To obtain the second equation, viz. $3x = 381$, we

collect all the terms containing the unknown quantity into one term.

We then divide this equation by 3, to find the value of x; for if $3\,x$ be equal to 381, x is evidently equal to $\frac{1}{3}$ of 381; that is, *we divide the terms of the equation by the coefficient of the unknown quantity.*

In general, to reduce an equation, *when the unknown quantity is multiplied by any number, we must divide all the other terms of the equation by that number.*

This principle must be observed, when the unknown quantity occurs in more than one term, and is multiplied by letters instead of numbers.

2. Given the equation $a\,x + b\,x = 38$.

As the first member, $a\,x + b\,x$, is the product of $a + b$ multiplied by x, it may be expressed thus: $(a + b)\,x$; where $a + b$ is the coefficient of x.

$$a\,x + b\,x = 38$$
$$(a + b)\,x = 38$$
$$x = \frac{38}{a+b}.$$

Let us suppose that $a = 10$, and $b = 9$, and substitute these numbers for the letters, we shall then have

$$10\,x + 9\,x = 38$$
$$(10 + 9)\,x = 38$$
$$x = \frac{38}{10+9} = 2.$$

*3. Two men, A and B, trade in company, and

* The questions thus marked are referred to in a subsequent chapter.

gain \$141, of which B is to have twice as much as A. What is the share of each?

Let $x =$ A's share.
Then $2\,x =$ B's share.
And $x + 2\,x = 141$.

ANS. A's share, \$47; B's, \$94.

4. A man distributed 70 cents among four poor persons; giving the second twice, the third three times, and the fourth four times, as much as he gave the first. What did he give to each?

*5. Said a father to his son, "Our joint ages are 78 years, and I am 5 times as old as you." What were their ages?

Let $x =$ the son's age.
Then $5\,x =$ the father's age.
And $5\,x + x = 78$.

*6. Three men, A, B and C, trade in company, and gain \$696, of which B is to receive 3 times as much as A, and C as much as both A and B. What is the share of each?

Let $x =$ A's share.
Then $3\,x =$ B's share,
and $x + 3\,x$, or $4\,x =$ C's share.
And $x + 3\,x + 4\,x = 696$.

7. A farmer hired two men and a boy to do a certain piece of work; agreeing to pay one of the men 5 shillings, the other 4 shillings, and the boy 3 shillings, a day. When the work was finished, he paid them \$54. How many days were they employed?

Let $x =$ the number of days. The first man earns $5x$ shillings; the other, $4x$ shillings; and the boy $3x$ shillings; which, together, are equal to 324, the shillings in \$54.

$$5x + 4x + 3x = 324.$$

8. A farmer sold an equal number of oxen, cows and sheep, for \$632. For the oxen he received \$47 apiece; for the cows, \$25; and for the sheep, \$7. How many did he sell of each sort?

9. A merchant, failing in trade, owes to A, B, C and D, \$3597. To B he owes twice as much as to A; to C, as much as to A and B; and to D, as much as to B and C. How much does he owe to each of them?

10. A boy bought 2 oranges, 3 pears and 4 apples, for 22 cents. He gave as much for a pear as for 2 apples; and twice as much for an orange, as for a pear and an apple. What was the price of each?

* 11. The age of A is double that of B; the age of B is three times that of C; and the sum of all their ages is 140. What is the age of each?

* 12. A farmer sold a quantity of wood for 104 dollars; one half of it at \$6 per cord; the other half at \$7. How many cords did he sell?

Let $x =$ half the number of cords.
Then $6x + 7x = 104$.

13. A man bought three equal lots of hay for \$366: for the first lot he gave \$19 a ton; for the second lot, \$20; for the third, \$22. How many tons did he buy?

* 14. How long will it take three masons to lay 1800 bricks, if one can lay 50, another 60, and the third 70 bricks an hour?

15. A man left an estate of $60000, to be so divided between his widow, 3 sons, 2 daughters and a ward, that each daughter should receive twice as much as the ward, each son as much as the ward and a daughter, and the widow twice as much as each son. What was the share of each?

16. How long will it take two men to build 387 rods of wall, if one build 4 and the other 5 rods a day?

Let $x =$ the number of days.

* 17. Four brothers gained, in a year, $4755; of which B gained three times as much as A; C gained as much as A and B; and D gained as much as B and C. What sum was gained by each?

18. In how many hours will a cistern, containing 264 gallons, be emptied by 3 cocks; one of which discharges 2 gallons in 15 minutes; the second, 5 gallons in 30 minutes, and the third, 3 gallons in 45 minutes?

* 19. One man leaves New York for Boston, and travels 9 miles an hour; another man leaves Boston for New York, and travels 7 miles an hour. In how many hours will they meet, the cities being 224 miles apart?

20. Four boys, A, B, C and D, upon counting their money, found they all had $30; of which sum A's share was three times greater than B's; C's share was equal to B's, and one third of A's; and D's share was equal to A's, and half of C's. What was the share of each?

$$\text{Let } x = \text{B's share.}$$
$$\text{Then } 3\,x = \text{A's share,}$$
$$x + \frac{3x}{3}, \text{ or } 2\,x = \text{C's share,}$$
$$\text{and } 3\,x + \frac{2x}{2}, \text{ or } 4\,x = \text{D's share.}$$
$$\text{Therefore, } x + 3\,x + 2\,x + 4\,x = 30.$$

21. A asked B how much money he had; who replied, that if he had seven times as much, he could lend four times what he then had, and have $69 left. How much had he?

$$\text{Let } x = \text{B's money.}$$
$$\text{Then } 7\,x - 4\,x = 69,$$
$$\text{or } 3\,x = 69.$$

22. A father is seven, and a mother five, times as old as their son; and the difference of their ages is 16 years. How old is the son?

23. A man directed, in his will, that his property should be so divided, that his son should have three times as much as his daughter, and his widow twice as much as both her children; by which division she received $8235 more than the son. What was the share of each?

24. A farmer bought a horse, a chaise and a house, for $800. Now, he paid 4 times as much for the chaise as for the horse, and 5 times as much for the house as for the chaise. What was the price of each?

*25. Two men, A and B, travel the same way; A at the rate of 45 miles, and B 30 miles, a day. In how many days will they be 300 miles apart?

*26. If they were to travel in different direc

tions, in how many days would they be 300 miles apart?

27. A man has six children, whose united ages are 42 years; and the common difference of their ages is equal to the age of the youngest child. What are their several ages?

*28. A gentleman bought 3 kinds of wine, of each an equal quantity. For the first kind he paid 7*s.*, for the second 8*s.*, and for the third 10*s.*, a gallon; and the price of the whole was $50. How many gallons did he buy?

*29. A farmer employed two men to build 105 rods of wall; one of whom could build 4 rods, and the other 3 rods a day. How many days did they work?

30. A laborer, who spent every week as much as he earned in 2 days, saved 32 dollars in 4 weeks. What were his daily wages?

31. A farmer sold a quantity of cider for 84 dollars; one half of it at $3 per barrel, the other half at $4. How many barrels did he sell?

32. A man left an estate of $21546; one third of which he bequeathed to his widow, and directed the remainder to be so divided between his 2 sons and 2 daughters, that each son might receive as much as both the daughters. What was the portion of each?

33. Three travellers found a purse, containing $54; of which B secured three times as much as A, and C secured half as much as both of the others. What was the share of each?

SECTION III.

To remove Denominators.

1. A man received \$18 towards a debt, which was only $\frac{2}{3}$ of the sum due. What was the whole debt?

Let x denote the amount of the debt.

Then $\frac{x}{3}$ must be $\frac{1}{3}$, and $\frac{2x}{3}$ must be $\frac{2}{3}$ of it.

Therefore, $\frac{2x}{3} = 18$.

It has already been shown, [See Sec. I.] that, if all the terms of an equation be either multiplied or divided by the same quantity, the equality of the members will not be affected; therefore, to remove the denominator of the first term,

Multiply by 3; $2x = 54$.
Divide by 2; $x = 27$. ANS. \$27.

We first multiply the whole equation by 3, the denominator of the fraction, and then find the value of x as before. It has been shown, in a former chapter, [See Chap. VI. Sec. VI.] that *a fraction is multiplied by its denominator, when that denominator is removed.*

2. What number is that, $\frac{1}{2}$ and $\frac{3}{5}$ of which are 44?

Let x indicate the number. Then $\frac{x}{2}$ is half of it, and $\frac{3x}{5}$ three fifths of it.

$$\text{Therefore, } \frac{x}{2} + \frac{3x}{5} = 44.$$

$$\text{Multiply by 2; } x + \frac{6x}{5} = 88.$$

$$\text{Multiply by 5; } 5x + 6x = 440.$$

$$\text{Add the } x\text{'s; } 11x = 440.$$

$$\text{Divide by 11; } x = 40. \qquad \text{Ans. } 40.$$

We first free the equations from fractions, by multiplying all the terms by the denominators, one at a time, and then find the value of x as before.

3. What number is that, which, being increased by $\frac{1}{2}$, $\frac{3}{4}$, $\frac{2}{7}$ and $\frac{1}{14}$ of itself, becomes 146?

$$\text{Let } x = \text{the number.}$$

$$\text{Then, by the question, } x + \frac{x}{2} + \frac{3x}{4} + \frac{2x}{7} + \frac{x}{14} = 146.$$

$$\text{Multiply by 2; } 2x + x + \frac{3x}{2} + \frac{4x}{7} + \frac{x}{7} = 292.$$

$$\text{Multiply by 2; } 4x + 2x + 3x + \frac{8x}{7} + \frac{2x}{7} = 584.$$

$$\text{Multiply by 7; } 28x + 14x + 21x + 8x + 2x = 4088.$$

$$\text{Add the } x\text{'s; } 73x = 4088.$$

$$\text{Divide by 73; } x = 56.$$

Ans. 56.

Observe that we multiply the fractions $\frac{3x}{4}$ and $\frac{x}{14}$, by dividing their denominators by 2; and we multiply the fractions $\frac{8x}{7}$ and $\frac{2x}{7}$, by removing their denominators.

To free an equation from fractions, we multiply all the terms by all the denominators, taking them in such order as may be most convenient.

4. Says A to B, "If to my age $\frac{1}{2}$ and $\frac{3}{4}$ of my age be added, the sum will be 81; what is my age?"

Let x denote A's age.

Then $x+\frac{x}{2}+\frac{3x}{4}=81$, by the question.

* 5. If $\frac{2}{7}$ of a ship cost $4000, what is the whole ship worth?

6. A man, driving his geese to market, was met by another, who said, "Good morrow, master, with your hundred geese." Said he, "I have not a hundred; but if I had as many more, and half as many more, and two geese and a half, I should have a hundred." How many had he?

$$x+x+\frac{x}{2}=100-2\tfrac{1}{2}.$$

7. A post, standing in a pond, is $\frac{1}{3}$ of its length under water, and $\frac{1}{4}$ above water, it being 14 feet from the top of the post to the bottom of the pond. What is the whole length of the post?

8. What number is that, whose sixth part exceeds its eighth part by 20?

Let $x=$ the number.

Then $\frac{x}{6}-\frac{x}{8}=20$.

* 9. A man, having spent three fifths of his estate, had $978 left. How much had he at first?

10. A man spent $\frac{1}{3}$ of his life in England, $\frac{1}{4}$ of it in Scotland, and the remainder of it, which was 20 years, in the United States. What was his age?

11. What number is that, $\frac{1}{3}$ of which is greater than $\frac{2}{7}$ of it by 21?

12. A man sold 75 bushels of wheat to two persons; to one, $\frac{1}{4}$, and to the other, $\frac{3}{8}$ of all he had. How many bushels had he?

13. A man gave to three poor persons \$6; to the first, $\frac{1}{3}$, to the second, $\frac{1}{4}$, and to the third, $\frac{1}{6}$ of all the money he had in his pocket. How much had he?

14. A and B divide \$320 between them, of which B has three times and $\frac{4}{7}$ as much as A. How much has each?

15. A says to B, "Your age is twice and $\frac{3}{5}$ of my age, and the sum of our ages is 54 years." What is the age of each?

* 16. If you divide \$50 between two persons, giving one $\frac{3}{7}$ as much as the other, what will be the share of each?

17. A stranger in Boston spent, the first day, $\frac{1}{3}$ of the money he brought with him; the second day, $\frac{1}{4}$; and the third day, $\frac{1}{5}$; when he had only \$26 left. How much money did he bring?

18. A man, having invested $\frac{3}{5}$ of his property in bank stock, by which he lost $\frac{2}{3}$ of the sum invested, had stock worth \$723 remaining. How much property had he at first?

19. A merchant retired from business when he had passed $\frac{9}{10}$ of his life, during $\frac{2}{3}$ of which period he had been engaged in trade, having commenced at the age of 21. At what age did he die?

20. The age of A is $\frac{1}{2}$ that of B, and the age of C is $\frac{1}{3}$ that of A, and the sum of all their ages is 120 years. What is the age of each?

21. A farmer wishes to mix 90 bushels of proven

der, consisting of barley, oats and corn, so that there may be $\frac{3}{5}$ as many bushels of oats as of corn, and $\frac{1}{3}$ as many bushels of barley as of oats. How many bushels of each sort must he use?

22. Three boys spent 98 cents for fruit. B spent $\frac{5}{8}$ as much as A, and C spent $\frac{1}{5}$ as much as B. What did each spend?

23. If you divide \$75 between two men, in the proportion of 3 to 2, what will each man receive?

One man will receive 2 dollars as often as the other receives 3; that is, one will receive $\frac{2}{3}$ as much as the other.

$$\text{Let } x = \text{the share of one.}$$

$$\text{Then } \frac{2x}{3} = \text{the share of the other.}$$

$$\text{And } x + \frac{2x}{3} = 75.$$

* 24. Divide 84 into two numbers, which shall be to each other as 7 to 5.

25. What two numbers are to each other as 4 to 9, their sum being 91?

* 26. Three men trade in company, and gain \$780. A put in \$2 as often as B put in \$3 and C put in \$5. What part of the gain must each one receive?

$$\text{Let } x = \text{A's share.}$$

$$\text{Then } \frac{3x}{2} = \text{B's share,}$$

$$\text{and } \frac{5x}{2} = \text{C's share.}$$

27. Divide 234 into 4 numbers, which shall be to each other in the proportion of 5, 6, 7 and 8.

28. Divide 36 into three such parts, that $\frac{1}{2}$ of the

first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall be equal to each other.

29. Four towns are situated in the order of the four letters, A, B, C, D. The distance from A to D is 102 miles. The distance from A to B is to the distance from C to D as 2 to 3; and $\frac{1}{4}$ of the distance from A to B, added to $\frac{1}{2}$ the distance from C to D, is 3 times the distance from B to C. How far are the towns apart?

30. Two young men began to trade at the same time, the capital of A being to that of B as 7 to 6. The first year, A gained a sum equal to $\frac{1}{5}$ of his capital, and B lost $\frac{1}{6}$ of his. The second year, A lost $\frac{3}{8}$ of what he then had; and B's gain was to what remained of his original capital, as 2 to 5. At the beginning of the third year, the two had \$2450. What was the original capital of each?

31. A gentleman, by his will, divided his property equally between his son and daughter; and the son, having spent $\frac{2}{3}$ of his portion, had \$7268 left. What was the whole amount of property left?

32. A man, being asked the age of his daughter, replied, "My age is to hers as 4 to 1, and her mother's age is to mine as 7 to 8, and the sum of all our ages is 102 years." What was the daughter's age?

33. A trader, having increased his capital by $\frac{1}{4}$ of itself, lost $\frac{1}{3}$ of what he then had; he afterwards gained a sum equal to $\frac{1}{2}$ of the remainder, when he was worth \$3935. What was his capital?

SECTION IV.

Transposition.

1. If you divide \$36 between two persons, in such a manner that B shall have \$12 more than A, what is the share of each?

Let $x =$ A's share.

Then $x + 12 =$ B's share.

A. And $x + x + 12 = 36$, by the question;

B. or $2x + 12 = 36$, by adding the x's.

In the last equation, we find that $2x + 12 = 36$; that is, $2x$ is 12 less than 36. We may free the first member from known quantities, by subtracting 12 from each side of the equation. We shall then have

C. $2x + 12 - 12 = 36 - 12.$

But as the $+ 12$ and $- 12$, in the first member, cancel each other, they may be omitted; thus,

D. $2x = 36 - 12;$

E. or $2x = 24$, by subtraction.

F. $x = \frac{24}{2} = 12$, A's part;

and $x + 12 = 24$, B's part.

Compare equations B and D; whence it appears, that $+ 12$ is transferred from the first to the second member of the equation, the sign being changed from $+$ to $-$.

Therefore, *to remove a positive quantity, produces the same result as to annex a negative quantity of the same value.*

The removing of a term from one side of an equation to the other, is called Transposition.

2. A trader has bills against A and B for $187, the bill of A being $25 more than that of B. What is the amount of each?

Let x denote B's bill.
Then $x + 25 =$ A's bill.
And $x + x + 25 = 187$, by the question.

* 3. A spent $87 more than B, and they both spent $325. How much was spent by each?

4. A gentleman gave to two beggars 67 cents, giving to the second 13 cents less than to the first. How many cents did each receive?

Let $x =$ the cents given to the first.
Then $x - 13 =$ the cents given to the second.
A. And $x + x - 13 = 67$, by the question;
B. or $2x - 13 = 67$, by addition of x's.

As $2x - 13 = 67$, it is evident that $2x$ is greater than 67 by 13. If we add 13 to each side of the equation, we shall have

C. $2x - 13 + 13 = 67 + 13.$

Since -13 and $+13$ cancel each other, in the first member, we have

D. $2x = 67 + 13;$
E. or $2x = 80$, by addition.
F. $x = \frac{80}{2} = 40$, the share of the first;
and $x - 13 = 27$, the share of the second.

Compare equations B and D, as before. It appears

that -13 is transferred from the first to the second member, the sign $-$ being changed to $+$.

Whence it appears, that *to remove a negative quantity, has the same effect as to add a positive quantity of equal value.*

5. Three men, A, B and C, trade in company, upon a capital of \$3981; of which B furnished \$337 more, and C \$181 less, than A. What was the share of each?

6. A man left an estate of \$9931, to be divided between his widow, son and daughter, in such a manner, that the son should have \$522 more than the daughter, and \$592 less than his mother. Required the portion of each.

7. Divide 42 into four such parts, that the first shall be 5 more than the second, 8 less than the third, and 9 more than the fourth.

* 8. An express had been travelling 5 days, at the rate of 60 miles a day, when another was despatched after him, who travelled 75 miles a day. In how many days did the latter overtake the former?

Let $x =$ the number of days. Then the second express travels $75\,x$ miles; and the first express travels $60\,x$ miles, after the departure of the second, and was 300 miles (60×5) in advance of him when he started. Now, as they both travel the same distance, we have

$$\text{A. } 75\,x = 60\,x + 300.$$

Here, the unknown quantity is found on both sides of the equation; *but all the terms containing the unknown quantity must be brought into one mem-*

ber, and all those containing known quantities, into the other.

If any commodity to be weighed, as coffee, for instance, were mixed with the *weights* of a grocer, although his scales might be perfectly balanced, it is evident that he could not determine how many pounds of coffee they contained. His only way would be, to collect all the coffee into one of the scales, and all the weights into the other. Now, an equation is not unlike the grocer's scales: the commodity to be weighed is the unknown, and the leaden weights, the known quantity.

If 60 x be subtracted from each side of the equation, we shall obtain

$$\text{B.}\quad 75\,x - 60\,x = 60\,x + 300 - 60\,x.$$

If we cancel $+ 60\,x$ and $- 60\,x$, in the second member, the equation will be

$$\text{C.}\quad 75\,x - 60\,x = 300;$$

$$\text{or } 15\,x = 300, \text{ by subtraction.}$$

$$x = 20.$$

ANS. 20 days.

Compare equations A and C, where a term containing the unknown quantity, 60 x, is removed from the second to the first member; the sign $+$ being changed to $-$.

9. The water had been flowing from a full cistern 6 hours, at the rate of 12 gallons an hour, when a pipe was conducted into it, which restored 21 gallons an hour. In how many hours was the cistern full again?

10. A man, being asked his age, answered, that if

his father's age, which was 52 years, were added to three times his own, the sum would be five times his age. How old was he?

* 11. A father is three times as old as his son; but in 20 years he will be only twice as old. What is the age of each?

12. When a boy would buy a certain number of oranges at 6 cents apiece, he found they would come to 12 cents more than he had; he, therefore, bought the same number at 5 cents each, and had 6 cents left. How many oranges did he buy?

* 13. Divide 64 into two such parts, that five times the first shall be equal to three times the second.

Let $x =$ the first part.
Then $64 - x =$ the second.
And $5x = (64 - x) \times 3$, by the question;
A. or $5x = 192 - 3x$.

It appears that $5x$ is not equal to 192 by $3x$. If we add $3x$ to each side of the equation, we have

B. $5x + 3x = 192 - 3x + 3x$.

And if we cancel the $-3x$ and $+3x$, in the second member, the equation becomes

C. $5x + 3x = 192$;
or $8x = 192$, by addition.
$x = 24$, the first part;
and $64 - x = 40$, the second part.

Observe, in the equations A and C, that $-3x$ is removed from the second member to the first, the sign $-$ being changed to $+$.

From the preceding examples and observations we derive the following principle: *Any term may be transposed from one member of an equation to the other, if the sign be changed.*

14. A pole is 4 feet in the ground, $\frac{1}{3}$ of its whole length under water, and $\frac{1}{2}$ above water. Required its length.

15. The head of a fish weighs 8 lbs.; his tail weighs as much as his head and half his body, and his body weighs as much as his head and tail. What is the weight of the fish?

Let $x =$ the weight of the body,

and $\frac{x}{2} + 8 =$ the weight of the tail.

Then $x = \frac{x}{2} + 16$.

16. A man, being asked his own and his wife's age, said, that his youngest child was 4 years old; that the age of his wife was twice the age of the child and $\frac{3}{4}$ of his own age; and that his own age was equal to the united ages of his wife and child. How old were they?

* 17. A merchant has wines at 9 shillings, and at 13 shillings, per gallon; and he would make a mixture of 100 gallons, that shall be worth 12 shillings per gallon. How many gallons of each must he take?

* 18. How many gallons of wine, at 9 shillings a gallon, must be mixed with 20 gallons at 13 shillings, that the mixture may be worth 10 shillings a gallon?

19. A merchant, having mixed 10 gallons of wine, at 8 shillings a gallon, with 25 gallons at 10 shillings,

wishes to add as much wine at 15 shillings as shall make the whole mixture worth 2 dollars a gallon. How many gallons must he take?

20. How many gallons of water must be mixed with 35 gallons of wine, at 9 shillings, and 45 gallons at 13 shillings, a gallon, that the whole mixture may be worth 10 shillings a gallon?

21. A clerk spends $\frac{2}{3}$ of his salary for board and lodging, $\frac{2}{3}$ of the remainder in clothes, and saves $150 per annum. What is his salary?

22. What is that number, $\frac{1}{3}$ and $\frac{1}{4}$ of which is 35 more than its sixth part?

*23. Two men, A and B, have each $80. A spends $5 more than twice as much as B, and has then half as much as B, wanting $13. How much did each spend?

24. Divide 84 into two such parts, that if $\frac{1}{2}$ of the less be subtracted from the greater, and $\frac{1}{8}$ of the greater be subtracted from the less, the remainders shall be equal.

Let $x =$ the greater number,
and $84 - x =$ the less.

Then $x - \frac{84-x}{2}$ is $\frac{1}{2}$ the less subtracted from the greater; and $84 - x - \frac{x}{8}$ is $\frac{1}{8}$ of the greater subtracted from the less; and, by the terms of the question, their values are equal.

$$x - \frac{84-x}{2} = 84 - x - \frac{x}{8}.$$

$$2x - 84 + x = 168 - 2x - \frac{x}{4}.$$

Ans. 48 and 36.

Observe that the sign —, before the fraction $\frac{84-x}{2}$, affects the value of the *whole* fraction, and not any particular term of it. We are not required to subtract half of 84 from the preceding term, but only half of the excess of 84 above x. Therefore, if the sign — is prefixed to 84, when the denominator is removed, the sign — before x must be changed to $+$; otherwise, too much, by the value of x, would be taken from the first term of the equation. Were the sign $+$ before the fraction $\frac{84-x}{2}$, it would not be necessary to change the sign before x from — to $+$. [See Chap. VI. Sec. III.]

25. Divide 72 into two unequal numbers, so that, if $\frac{3}{4}$ of the less be subtracted from the greater, the remainder may be equal to $\frac{2}{3}$ of the number which remains, after the excess of the greater above the less is subtracted from the less.

26. Two clerks, A and B, sent ventures in different vessels, A's being worth only $\frac{3}{5}$ as much as B's. A gained and B lost \$23; then $\frac{7}{8}$ of B's returns, subtracted from A's, would leave $\frac{1}{2}$ of the value of A's venture. How much did each send?

27. A gentleman bought a watch and chain for \$160. If $\frac{3}{4}$ of the price of the watch be subtracted from 6 times the price of the chain, and $\frac{5}{12}$ of the price of the chain be subtracted from twice the price of the watch, the remainders will be equal. What was the price of each?

28. A person, being asked the time of day, replied, "If to the time from noon be added its $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{6}$,

the sum will be equal to the time till midnight. Required the hour.

29. There are two numbers in the proportion of 3 to 4; but if 24 be added to each of them, the sums will be in the proportion of 4 to 5. What are the numbers?

* 30. What number is that which, being added to 5, and also multiplied by 5, the product shall be 4 times the sum?

31. A man, having spent $10 more than $\frac{1}{3}$ of his money, had $15 more than $\frac{1}{2}$ of it left. How much had he?

* 32. What number is that, which, being divided by 16, will amount to 84, when the quotient, dividend and divisor are added together?

SECTION V.

Questions for Practice.

1. What number is that, to which if there be added $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of itself, the sum will be 50?

2. A person, upon being asked his age, replied, that $\frac{1}{3}$ exceeded $\frac{1}{4}$ part of it by 5 years. What was his age?

3. A trader gave three checks, amounting to $94; the first for $\frac{1}{3}$, the second for $\frac{1}{4}$, the third for $\frac{1}{5}$, of the money he had in the bank. How much had he?

4. To find two such numbers, in the proportion of

2 to 1, that, if 4 be added to each, they will be in proportion of 3 to 2.

5. In the composition of a certain quantity of gunpowder, $\frac{2}{3}$ of the whole and 10 pounds were nitre; $\frac{1}{6}$ of the whole, wanting $4\frac{1}{2}$ pounds, was sulphur; and the charcoal was $\frac{1}{7}$ of the nitre, wanting 2 pounds. How many pounds were there?

*6. A person, being asked the time of day, answered, that the time past from noon was equal to $\frac{4}{5}$ of the time to midnight. What was the hour?

$$\text{Let } x = \text{the hour.}$$

$$\text{Then } 12 - x = \text{time to midnight.}$$

$$\text{And } x = \frac{48-4x}{5}.$$

7. A privateer, running at the rate of 10 miles an hour, discovers a ship 18 miles off, making way at the rate of 8 miles an hour. In how many hours will the ship be overtaken?

8. Required two numbers, which are to each other as 3 to 2; and whose sum equals $\frac{1}{6}$ part of their product.

Let x denote the larger number, and the smaller number will be $\frac{2x}{3}$, and $x + \frac{2x}{3} =$ their sum.

Now, $\frac{2x}{3} \times \frac{x}{1} = \frac{2x^2}{3}$, which is the product of the two numbers; and $\frac{1}{6}$ part of $\frac{2x^2}{3}$ is $\frac{2x^2}{18}$, or $\frac{x^2}{9}$.

$$\text{Then } \frac{x^2}{9} = x + \frac{2x}{3}, \text{ by the question.}$$

$$x^2 = 9x + 6x, \text{ by removing the denominators.}$$

This is an equation of the second degree, as it contains the second power of the unknown quantity,

namely, x^2; but if we divide all the terms by x, we shall obtain

$$x = 9 + 6,$$

which is an equation of the first degree.

ANS. 15 and 10.

9. Divide 48 into two such parts, that the one part may be three times as much above 20 as the other wants of 20.

10. A father leaves $1600, to be divided between his widow, son and daughter, in such a manner that the widow is to have $200 more than the son, and the son $100 more than the daughter. What is the share of each?

11. A man leaves $11000, to be divided between his widow, two sons, and three daughters. By his will the mother is to receive twice as much as one of the sons; and each son is to receive twice as much as a daughter. How much is each of them to receive?

12. An estate is divided between three men in such a manner, that A receives $1000 less than $\frac{1}{2}$, B $800 less than $\frac{1}{3}$, and C $600 less than $\frac{1}{4}$ of the whole. What is the value of the whole estate, and what is the share of each individual?

13. A father leaves his property to four sons, who share it in the following manner: A has $3000 less than $\frac{1}{2}$; B has $1000 less than $\frac{1}{3}$; C has $\frac{1}{4}$; and D has $600 more than $\frac{1}{5}$ of the property. What is the whole amount bequeathed? and the share of each of the sons?

* 14. A father, being asked the age of his daughter, replied, that, if her age were multiplied by 5, the

product would be as much less than 40, as her age was less than 12. What was her age?

* 15. Divide 76 into two such parts, that the quotient of the greater, divided by the less, may be 37.

$$\text{Let } x = \text{less number.}$$
$$\text{Then } 76 - x = \text{the greater.}$$
$$\text{And } \frac{76-x}{x} = 37.$$

* 16. A man has two silver cups, having but one cover for both. The first cup weighs 12 ounces; and when it is covered, it weighs twice as much as the other cup; but if the second cup be covered, it weighs three times as much as the first. Required the weight of the cover and of the second cup.

* 17. If a certain number be divided by 7, the sum of the dividend, divisor and quotient, will be 71. What is the number?

18. What is that number, of which $\frac{1}{7}$ is as much smaller than 65, as twice the number is greater than 640?

19. Two persons, A and B, are 320 miles apart, and travel towards each other; A at the rate of 9 miles an hour, and B at the rate of 7 miles. In what time will they meet? How many miles does each travel?

20. Two brothers, A and B, had the same income. A spent all of his, and $\frac{1}{7}$ more; B saved $\frac{1}{5}$ of his. At the end of 10 years, B paid A's debts, and had $160 left. What was their income?

21. A farmer planted corn in 4 fields. The third produced 9 bushels more than the fourth; the second,

12 bushels more than the third; the first, 18 bushels more than the second; and the whole produced 6 bushels more than 7 times as much as the fourth. What was the whole number of bushels?

*22. Two men have equal sums of money. One having spent $39, and the other $93, the one has but half as much left as the other. How much had each?

23. What is that number, which being added to 24 and 36, the sums will be to each other as 7 to 9?

*24. A laborer receives 3*s*. 6*d*. every day he works, and forfeits 9*d*. every day he is idle. At the end of 24 days, there is due to him the sum of £3 2*s*. 9*d*. How many days was he idle?

*25. Two persons, A and B, have each an annual income of $400. A spends, every year, $40 more than B; and, at the end of 4 years, they both together save a sum equal to the income of either. What do they spend annually?

26. A gentleman leaves $315, to be divided among four servants in the following manner: B is to receive as much as A, and $\frac{1}{2}$ as much more; C is to receive as much as A and B, and $\frac{1}{3}$ as much more; D is to receive as much as the other three, and $\frac{1}{4}$ as much more. What is the share of each?

27. What number is that, which being multiplied by 4, and 30 subtracted from the product, and being divided by 4, and 30 added to the quotient, the sum and difference shall be equal?

28. A person, being asked his age, replied, "If $\frac{3}{8}$ of my age be multiplied by $\frac{1}{9}$ of my age, the product will be equal to my age." What was his age?

29. Two numbers are to each other as 2 to 3; but if 50 be subtracted from each, one will be $\frac{1}{2}$ of the other. What are the numbers?

* 30. A waterman went down a river and returned again in 6 hours. Now, *with* the stream, he can row 9 miles an hour; but, *against* it, he can make a headway of only 3 miles an hour. How far did he go?

31. In a mixture of wine and cider, $\frac{1}{2}$ of the whole and 25 gallons was wine; and $\frac{1}{3}$ of the whole, wanting 5 gallons, was cider. Required the quantity of each in the mixture.

32. A gentleman visited several poor families. At the first house, he gave 3 shillings less than $\frac{1}{4}$ of all his money; at the second, 4 shillings less than $\frac{1}{3}$ of what he had left; at the third, $\frac{1}{4}$ of what he had left; at the fourth, 3 shillings less than $\frac{1}{2}$ of what he had left; at the fifth, 2 shillings more than $\frac{1}{3}$ of what he had left; and at the sixth, all he had remaining. Now, he gave the same sum to each family. How much did he leave at each house?

33. In a certain river is a post that stands 4 feet in the ground; at high water, $\frac{5}{6}$ of the remainder is covered, while only $\frac{1}{8}$ of the whole post appears above the surface of the water; but, at low water, the length above the surface is just equal to that which, at high water, is immersed. Required the length of the post, and the rise of the tide.

* 34. What number is that, to which if I add 13, and from $\frac{1}{13}$ of the sum subtract 13, the remainder shall be 13?

35. Three men, A, B and C, build 318 rods of

wall: A builds 7 rods, B 6 rods, and C 5 rods a day, B works twice as many days as A, and C works $\frac{1}{2}$ as many days as both A and B. How many days does each work?

36. A farmer has his sheep in four pastures: in the first, $\frac{1}{3}$ of his flock; in the second, $\frac{1}{4}$; in the third, $\frac{1}{6}$; and in the fourth, 18 sheep. How many sheep has he?

* 37. If a man fills a certain chest with corn, at 5*s.* a bushel, he will spend all his money; but if he fills it with oats, at 3*s.* 6*d.* a bushel, he will have £1 4*s.* left. How many bushels does the chest hold?

38. Three merchants, A, B and C, engage in a speculation, by which they gain $960. A put in $3 as often as B $7, and C $5. What is each man's share of the gain?

* 39. Says John to William, "I have three times as many marbles as you." "Yes," says William; "but if you will give me 20, I shall have 7 times as many as you." How many has each?

40. A woman sells eggs at 5 cents a dozen more than apples; and 8 dozen of eggs are worth as much as $13\frac{5}{7}$ dozen of her apples. What is the price of each?

* 41. Two travellers found some five-dollar bills in the road, of which A secured twice as many as B; but had B secured 5 more of the bills, he would have had 3 times as much money as A. How much did each find?

42. A man's age, when he was married, was to that of his wife as 6 to 5; and after they had been mar-

ried 8 years, her age was to his as 7 to 8. What were their ages when they were married?

43. A gentleman gave $44 more for his chaise than for his horse. Now, if $\frac{4}{7}$ of the price of the horse be subtracted from the price of the chaise, the remainder will be the same as if $\frac{2}{3}$ of the excess of the price of the horse above $84 be subtracted from the price of the horse. What did he give for the horse?

44. A hare is 50 leaps before a hound, and takes 4 leaps to the hound's 3; but 2 of the hound's leaps are equal to 3 of the hare's. How many leaps must the hound take to catch the hare?

45. A man and his wife usually drank a cask of beer in 12 days; but when the man was from home, it lasted his wife 30 days. How many days would it last the man alone?

Let x denote the number of days it would last the man alone. Now, as the cask would last the man and woman together 12 days, in one day *the two* would drink $\frac{1}{12}$ of it. Also, as it would last the *woman* 30 days, in one day she would drink $\frac{1}{30}$ of it; and $\frac{1}{12} - \frac{1}{30}$ expresses the proportion of the cask which the *man* drank in a day. But if he drank the whole of it in x days, in one day he drank $\frac{1}{x}$ part of it.

$$\text{Therefore, } \frac{1}{12} - \frac{1}{30} = \frac{1}{x}.$$

* 46. A and B together can build a piece of wall in 8 days; and, with the assistance of C, they can build it in 5 days. In how many days could C build it alone?

*47. A cistern has two cocks, one of which will empty it in 7 hours, the other in 9 hours. How long will it take both to empty it?

Let x denote the number of hours required to empty the cistern.

The two cocks will discharge $\frac{1}{7}+\frac{1}{9}$ of the water in an hour. Also, if they will empty the cistern in x hours, in one hour they will discharge $\frac{1}{x}$ part of the water.

$$\text{Therefore, } \frac{1}{7}+\frac{1}{9}=\frac{1}{x}.$$

48. If a reservoir can be exhausted by one engine in 7 hours, by another in 8 hours, and by a third in 9 hours, in what time will it be exhausted, if they are all worked together?

*49. A reservoir can be filled by two hose companies in 12 hours, and by one of them alone in 20 hours. In what time could the other fill it?

50. In an orchard of fruit-trees, $\frac{1}{2}$ of them bear apples, $\frac{1}{4}$ of them pears, $\frac{1}{6}$ of them peaches, 7 trees bear cherries, 3 plums, and 2 quinces. How many trees are there in the orchard?

51. A boy, being asked his age, answered, that if $\frac{1}{2}$ and $\frac{1}{4}$ of his age and 20 years more were added to his age, the sum would be three times his age. How old was he?

*52. A father is 40 years old, and his son is 8. In how many years will the father's age be 3 times the son's?

53. Two travellers, A and B, find a purse with dollars in it. A takes out $2 and $\frac{1}{6}$ of what remains

and B takes out \$3 and $\frac{1}{6}$ of what remains; when they have equal shares. How much money did they find?

* 54. There is a certain number consisting of two digits or figures, and their sum is 6. If 18 be added to the number, the sum will consist of the same digits inverted. What is the number?

Since the number consists of two digits, the first must be in the place of *tens*. Let this be denoted by x. Then, since their sum is 6, the other digit must be $6 - x$; and the number is $10x + 6 - x$. Now, if 18 be added to the number, the digits will be inverted.

$$\text{Therefore, } 10x + 6 - x + 18 = 10(6 - x) + x,$$
$$\text{or } 10x + 6 - x + 18 = 60 - 10x + x.$$
$$10x - x + 10x - x = 60 - 6 - 18;$$
$$18x = 36.$$
$$\left.\begin{array}{r} x = 2, \\ \text{and } 6 - x = 4, \end{array}\right\} \text{the digits.}$$

The number is 24; and $24 + 18 = 42$.

55. Required the number, from which if 27 be subtracted, the digits of which it is composed will be inverted; the sum of the digits being 9.

56. A grocer bought a quantity of oats at the rate of 2 bushels for a dollar, and as many more at the rate of 3 bushels for a dollar; and he sold them 5 bushels for 3 dollars, by which he gained \$10. How many bushels did he sell?

Let $x =$ the bushels of each sort.

Then $\frac{x}{2}$ = the cost of the first purchase,

and $\frac{x}{3}$ = the cost of the second purchase.

$\frac{x}{2} + \frac{x}{3} + 10$ = the whole cost and profits.

As he sold 5 bushels for \$3, he sold each bushel for \$$\frac{3}{5}$; and, of course, he received for $2x$ bushels, that is, for x bushels of each sort, \$$\frac{6x}{5}$.

$$\text{Therefore, } \frac{6x}{5} = \frac{x}{2} + \frac{x}{3} + 10.$$

Ans. $54\frac{6}{11}$ bushels.

57. Two clerks, A and B, have the same income. A saves $\frac{1}{5}$ of his; but B, by spending \$80 a year more than A, at the end of 4 years finds himself \$220 in debt. What was their income?

58. After spending $\frac{1}{4}$ of my money, and $\frac{1}{5}$ of the remainder, I had \$96 left. How much had I at first?

59. A traveller spent $\frac{1}{3}$ of his money in Boston; $\frac{1}{7}$ of the remainder in Providence; $\frac{4}{5}$ of what was left in New York; $\frac{3}{8}$ of the balance in Philadelphia, and had \$80 left. How much had he at first?

60. Divide 26 into three such parts, that, if the first be multiplied by 2, the second by 3, and the third by 4, the products shall all be equal.

61. Divide 56 into two such parts, that, the larger being divided by 7, and the smaller by 3, the sum of their quotients may be 10.

62. A cistern has three cocks; the first will fill it in 5 hours, the second in 10 hours, and the third will empty it in 8 hours. In what time will the cistern be filled, if all the cocks are running together?

63. A school-boy, being asked his age, replied, that $\frac{3}{4}$ of his age multiplied by $\frac{1}{12}$ of his age, would give a product equal to his age. How old was he?

64. A person has a lease for 99 years; and, being asked how much of it had expired, he replied, that $\frac{2}{3}$ of the time past was equal to $\frac{4}{5}$ of the time to come. How many years had the lease run?

65. What number is there which may be divided into either two or three equal parts, and the continued product of those parts shall be equal?

66. A shepherd, driving a flock of sheep in time of war, meets with a company of soldiers, who plunder him of half his flock and half a sheep over; and a second, third and fourth company treat him in the same manner, each taking half the flock left by the last and half a sheep over, when but 8 sheep remained. How many sheep had he at first?

67. A gentleman has two horses, and a chaise worth \$150. Now, if the first horse be harnessed, the horse and chaise together will be worth twice as much as the second horse; but if the second horse be harnessed, they will be worth three times as much as the first horse. What is the value of each horse?

*68. Divide 54 into two such parts, that, if the greater be divided by 9, and the less by 6, the sum of the quotients shall be 7.

69. A farmer sells a quantity of corn, which is to the quantity left as 4 to 5. After using 15 bushels, he finds he has $\frac{1}{2}$ as much left as he sold. How many bushels had he at first?

70. Divide 84 into two such numbers, that the quo

tient of the greater, divided by their difference, may be four.

*71. A laborer agreed to work for a gentleman a year, for $72 and a suit of clothes; but at the end of 7 months, he was dismissed, having received his clothes and $32. What was the value of the clothes?

72. A laborer reaps 35 acres of wheat and rye For every acre of rye he receives 5 shillings; and what he receives for an acre of wheat, if it were 1 shilling more, would be to what he receives for an acre of rye as 7 to 3. For the whole he receives £13. How many acres are there of each sort?

*73. A man and his wife consumed a sack of meal in 15 days. After living together 6 days, the woman alone consumed the remainder in 30 days. How long would the sack last either of them alone?

74. A company of men, women and children consists of 75 persons. The number of the men exceeds that of the women by 5, and there are 13 more children than adults. Required the number of men women and children.

75. Three adventurers, A, B and C, bought 10170 acres of wild land. By the terms of the contract, B had 549 acres less than A, and C had 987 acres more than B. How many acres had each?

76. A man, being asked how much money he had, replied, "If you multiply my money by 4, add 60 to the product, divide the sum thus obtained by 3, and then subtract 45 from tne quotient, the remainder will be the number of dollars I have." How much money had he?

SECTION VI.

Two Unknown Quantities.

Elimination by Comparison.

1. A fruiterer sold to one person 6 lemons and 3 oranges for 42 cents; and to another, 3 lemons and 8 oranges for 60 cents. What was the price of each?

This question can be solved, without difficulty, by means of only one unknown quantity: the solution will be more simple, however, if two unknown quantities are used. *When the conditions of a question are such as require two or more unknown quantities, our first object is to obtain an equation involving but one unknown quantity.* This process is called *Elimination.* There are several methods of elimination, with which the learner should become familiar, as he may often find it convenient to use them all in the same operation.

Let $x =$ the price of a lemon,
and $y =$ the price of an orange.

A. Then $6x + 3y = 42$, by the first sale.

B. $6x = 42 - 3y$, by transposition.

C. $x = \frac{42-3y}{6}$, by division.

D Again, $3x + 8y = 60$, by the second sale

E. $3x = 60 - 8y$, by transposition.

F. $x = \frac{60-8y}{3}$, by division.

Now, as *things, which are equal to the same thing, are equal to each other,* we can form a new equation

from the value of x, as determined in equations C and F, which will contain but one unknown quantity

$$\text{G.}\quad \frac{42-3y}{6}=\frac{60-8y}{3}.$$

$$\text{H.}\quad 126-9y=360-48y, \text{ by multiplication.}$$

$$\text{I.}\quad 48y-9y=360-126, \text{ by transposition.}$$

$$\text{J.}\quad 39y=234, \text{ by uniting terms.}$$

$$\text{K.}\quad y=6, \text{ the price of an orange.}$$

By substituting this value of y in equation C, we have

$$x=\frac{42-(3\times 6)}{6}=\frac{42-18}{6};$$

$$\text{or } x=\tfrac{24}{6}=4, \text{ the price of a lemon.}$$

When a question involves two unknown quantities, its conditions must admit of two equations, or it cannot be solved. By this method of elimination, we *find the value of one of the unknown quantities in each of the equations, and make the expressions of its value, thus found, equal to each other.* An equation is thus obtained, involving but one unknown quantity.

*2. A gentleman has two silver cups, and a cover adapted to each, which is worth £10. If the cover be put upon the first cup, its value will be twice that of the second; but if it be put upon the second, its value will be three times that of the first. What is the value of each cup?

Let $x =$ the value of the first cup,
and $y =$ the value of the second cup.
Then $2y = x + 10$, by the question.

$$y=\frac{x+10}{2}.$$

Again, $y + 10 = 3x$, by the question.
$$y = 3x - 10.$$
And $3x - 10 = \frac{x+10}{2}$, by comparing values of y

ANS. First cup, £6; second, £8

*3. There are two numbers whose sum is 120; and if 4 times the less be subtracted from 5 times the greater, the remainder will be 150. Required the numbers.

4. If the greater be added to half the less of two numbers, the sum is 48; but if the less be added to half the greater, the sum is 42. What are the numbers?

5. A vintner has two sorts of wine, which, if mixed in equal parts, will be worth 15 shillings a gallon; but if 2 gallons of the first be mixed with 3 gallons of the second, a gallon of the mixture will be worth only 14 shillings. What is each sort worth per gallon?

6. If we add 7 to the numerator of a fraction, its value becomes 2; if we add 7 to its denominator, it becomes $\frac{1}{2}$. What is the fraction?

7. It is required to find two numbers, such that $\frac{1}{3}$ of the first and $\frac{1}{4}$ of the second shall be 87, and $\frac{1}{5}$ of the first and $\frac{1}{6}$ of the second shall be 55.

8. Says A to B, "6 years ago, your age was double mine; and, in 4 years, my age will be $\frac{3}{5}$ of yours." What is the age of each?

9. There is a number consisting of two figures; and, if the number be divided by the sum of the figures, the quotient is 4; but if the figures be inverted,

and the number divided by 1 more than their sum, the quotient will be 6. What is the number?

Let $x =$ the first figure,
and $y =$ the second figure.
Then $10\,x + y =$ the number.

As the first figure is in the place of *tens*, it must be multiplied by 10, to express its *local* value.

$$\left.\begin{aligned}\frac{10x+y}{x+y} &= 4\\ \frac{10y+x}{x+y+1} &= 6\end{aligned}\right\}\text{ by the conditions of the question.}$$

10. A farmer sold to one man 10 bushels of corn and 12 bushels of potatoes for 54 shillings; and to another, 2 bushels of corn and 4 bushels of potatoes for 14 shillings. What was the price of each per bushel?

11. A gentleman gave to his two sons, A and B, 9800 dollars. At the end of a year, A finds that he has spent $\frac{1}{6}$ of his share; but B, having spent only $\frac{1}{8}$ of his, has just as much left as his brother. What was the share of each?

12. Says A to B, "Give me 6 dollars, and I shall have 4 times as much as you." "Rather give me 3 dollars," says B, "and I shall have just as much as you." How many dollars has each?

*13. If I take 10 apples from A, he will still have twice as many as B; but if I give them to B, they will each have the same number. How many have they?

14. If you multiply the greater of two numbers by

2 and the less by 3, the sum of their products is 101. And if you divide the greater by 4 and the less by 5, the sum of their quotients is 10. Required the numbers.

Elimination by Substitution.

15. A draper sold a yard of broadcloth and 3 yards of velvet for 25 dollars; and afterwards he sold 4 yards of broadcloth and 5 yards of velvet for 65 dollars. What was the price of each per yard?

Let $x =$ the price of the broadcloth,
and $y =$ the price of the velvet.

A. Then $x + 3y = 25$, by the first sale.
B. and $4x + 5y = 65$, by the second sale.
C. $4x = 65 - 5y$, by transposition.
D. $x = \frac{65 - 5y}{4}$, by division.

Now, if we substitute the value of x for x itself in equation A, we shall obtain a new equation with but one unknown quantity.

E. $\frac{65 - 5y}{4} + 3y = 25$

$$65 - 5y + 12y = 100$$
$$7y = 100 - 65 = 35$$
$y = 5$, the price of the velvet.

D. $x = \frac{65 - 5y}{4} = \frac{40}{4} = 10$, the price of the broadcloth.

Either of the unknown quantities may be thus made

to disappear; but it will be found convenient to eliminate that quantity which is the least involved.

16. A builder paid 5 men and 3 boys 42 shillings for working a day; he afterwards hired 7 men and 5 boys a day for 62 shillings. What were the wages of each?

Let $x =$ the wages of a man,

and $y =$ the wages of a boy.

Then $5\,x + 3\,y = 42$, paid the first day.

$5\,x = 42 - 3\,y$, by transposition.

$$x = \frac{42 - 3\,y}{5}.$$

Again, $7\,x + 5\,y = 62$, paid the second day.

$\frac{294 - 21\,y}{5} + 5\,y = 62$, by substitution.

$$294 - 21\,y + 25\,y = 310$$

$$4\,y = 16$$

$y = 4s.$ a boy's wages.

$x = \frac{42 - 3\,y}{5} = \frac{42 - 12}{5} = 6s.$ a man's wages.

Let the following questions be solved in the same manner, namely: *Find the value of either of the unknown quantities in one of the equations, as if the other were known; and use the value thus found, instead of the quantity itself, in the other equation.* This method of making one of the unknown quantities disappear, is called elimination by *substitution.*

17. Says A to B, "Give me \$8, and I shall have twice as much money as you will have left; but if I give you \$6, my money will be equal to but $\frac{1}{4}$ of yours." How much has each?

18. A gentleman has two horses, and a chaise worth

$250. If the first be harnessed, the horse and chaise will be worth twice as much as the second horse; but if the second be harnessed, they will be worth three times as much as the first horse. What is the value of each horse?

19. A merchant bought two lots of flour for $576; the first lot for $5, and the second for $6, per barrel. He then sold $\frac{5}{8}$ of the first lot and $\frac{4}{7}$ of the second for $353, by which he gained $11. How many barrels were there in each lot?

20. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{1}{2}$; but the denominator being doubled, and the numerator increased by 7, the value becomes 1?

21. A teacher, being asked the dimensions of his school-room, answered, that if it were 5 feet broader and 3 feet longer, the floor would contain 422 feet more; but if it were 3 feet broader and 5 feet longer, the floor would contain 400 feet more. What were the dimensions of the room?

$$\text{Let } x = \text{the length of the room,}$$
$$\text{and } y = \text{the breadth.}$$

Now, to find the area, we must multiply the length by the breadth. Consequently, $x\,y =$ the area of the floor.

$$\text{Then } (x + 5) \times (y + 3) = x\,y + 400,$$
$$\text{and } (x + 3) \times (y + 5) = x\,y + 422.$$

22. If 6 feet were added to each side of a hall, the breadth would be to the length as 6 to 7; but if 6 feet were taken from each of the sides, they would be to

each other as 4 to 5. Required the dimensions of the hall.

23. A farmer has 86 bushels of wheat at 4*s*. 6*d*. a bushel, with which he wishes to mix rye at 3*s*. 6*d*. a bushel, and barley at 3*s*. a bushel, so as to make 136 bushels, that shall be worth 4*s*. a bushel. How much rye and barley must he take?

24. A merchant put 13 crates and 33 bales of goods into a warehouse, which was all that it would hold. After he had removed 5 crates and 9 bales, he found that the house was two thirds full. How many crates or bales would it take to fill it?

25. If you multiply the greater of two numbers by 3 and the less by 4, the difference of their products is 48; but if you divide the greater by 4 and the less by 3, the sum of their quotients will be 14. What are the numbers?

26. A gentleman, having a quantity of gold and silver coins, finds that 24 pieces of gold and 40 pieces of silver, will pay a certain debt; of which 5 pieces of gold and 15 pieces of silver, will pay $\frac{1}{4}$ part. How many pieces of gold, and how many of silver, will pay the whole debt?

27. In an election, the two candidates received 1384 votes; but if the successful candidate had received but half his number of votes, and three times as many as he received had been given for the other, the whole number of votes would have been 2102. How many votes were given for each?

28. The length of a certain garden, which contains 128 square rods, is twice as great as its width; and

if the garden were 4 rods longer, it would contain an acre. Required its length and width.

Elimination by Addition and Subtraction.

29. A man bought 3 bushels of wheat and 5 bushels of rye for 38 shillings; he afterwards bought 6 bushels of wheat and 3 bushels of rye for 48 shillings. What did he give a bushel for each?

Let $x =$ the price of a bushel of wheat,
and $y =$ the price of a bushel of rye.

A. Then $3\,x + 5\,y = 38$, by the first purchase,
B. and $6\,x + 3\,y = 48$, by the second purchase.

If we multiply all the terms of equation A by 2, [See Sec. I. of this Chap.] we shall have

C. $6\,x + 10\,y = 76.$

Now, if we subtract equation B from equation C, the remainder will be a new equation, containing only one unknown quantity, whose value may be found as before. Thus,

C. $6\,x + 10\,y = 76$
B. $6\,x + 3\,y = 48$

D. * $7\,y = 28$, by subtraction.
$y = 4$, the price of the rye.

By substituting this value of y, in *either* of the above equations, we shall obtain the value of x.

$$\text{A.}\quad 3x + (5 \times 4) = 38$$
$$3x = 38 - 20 = 18$$
$$x = 6, \text{ the price of the wheat.}$$

The student will now understand why equation A is multiplied by 2. Having determined to eliminate x, I wish to make the coefficients of the terms con taining x, in the equations A and B, equal, that they might cancel each other when subtracted. It matters not which of the unknown quantities is eliminated in this way.

30. A boy bought 7 oranges and 5 lemons for 55 cents; and afterwards let one of his companions have 4 oranges and 3 lemons for 32 cents, which was their cost. What was the price of each?

Let $x =$ the price of an orange,
and $y =$ the price of a lemon.

A. Then $7x + 5y = 55,$ } by the conditions of the
B. and $4x + 3y = 32,$ } question.

We will first eliminate y. As we cannot multiply either equation by any number, which will make the coefficients of the terms containing y alike, we must multiply both equations by such numbers as will produce this result.

C. $21x + 15y = 165$, by multiplying equation A by 3.
D. $\underline{20x + 15y = 160}$, by multiplying equation B by 5.
$x \quad * \quad = \quad 5$, by subtraction.
B. $(4 \times 5) + 3y = \quad 32$, by substitution.
$3y = 32 - 20 = 12$; and $y = 4$.

ANS. Oranges, 5; lemons, 4 cents.

31. A gentleman paid for 6 pair of boots and 8 pair of shoes \$52; he afterwards paid for 3 pair of boots and 7 pair of shoes \$32. How much were the boots and shoes a pair?

Let $x =$ the price of the boots,
and $y =$ the price of the shoes.

A. Then $6x + 8y = 52$, } by the conditions of the
B. and $3x + 7y = 32$, } question.
C. $3x + 4y = 26$, by dividing equation A by 2.

E. * $3y = 6$, by subtracting equation C from B.

$y = 2$, the price of a pair of shoes.

C. $3x + (4 \times 2) = 26$, by substitution.

$3x = 26 - 8 = 18$,

and $x = 6$, the price of a pair of boots.

The equation A is divided by 2, to make the coefficient of x, the quantity to be eliminated, equal to the coefficient of the same quantity in equation B.

32. If twice A's money be subtracted from 3 times B's, the remainder is \$38; but if twice B's money be subtracted from 3 times A's, the remainder is \$83. How much has each?

Let $x =$ A's money,
and $y =$ B's money.

A. Then $3y - 2x = 38$, } by the conditions of the
B. and $3x - 2y = 83$, } question.

If we multiply equation A by 3, and equation B by 2, the coefficients of the terms containing x will be alike.

C. $9y - 6x = 114$, by multiplying equation A by 3.
D. $-4y + 6x = 166$, by multiplying equation B by 2.

$5y \quad * = 280$, by addition,
and $y = 56$, B's money.
B $3x - (2 \times 56) = 83$, by substitution.
$3x = 83 + 112 = 195$,
and $x = 65$, A's money.

As the terms of the equations C and D, containing x, have unlike signs, they are cancelled by *addition.*

From these operations may be derived the following RULE for removing one of the unknown quantities:

Having determined which of the unknown quantities you will eliminate, make the coefficients of the terms, containing that quantity, the same in both equations, either by multiplication or division.

If the signs of these terms are unlike, add both equations together: if they are alike, subtract the smaller from the larger equation.

When the student does not readily find a proper multiplier, he can multiply each equation by the coefficient of that term, in the other, which contains the unknown quantity to be removed.

33. Says A to B, "$\frac{1}{3}$ of the difference of our money is equal to yours; and, if you give me \$2, I shall have 5 times as much as you." How much has each?

34. There is a certain number consisting of two figures; and if 2 be added to the sum of its digits, the amount will be 3 times the first digit; and if 18

be added to the number, the digits will be inverted. What is the number?

35. A man has money in dwo drawers, and \$25 in his purse. Now, if he put his purse into the first drawer, it will contain $\frac{5}{7}$ as much as the second; but if he put his purse into the second drawer, the sum in the first will be to the sum in the second as 5 to 13. How much is there in each drawer?

36. Two clerks, A and B, send ventures, by which A gained \$20, and B lost \$50, when the former had twice as much as the latter; but had B gained \$20, and A lost \$50, then B would have had 4 times as much as A. What sum was sent by each?

37. A farmer, having mixed a certain quantity of barley and oats, found that, if he had mixed 6 bushels more of each, he would have put into the mixture 7 bushels of barley for every 6 of oats; but if he had mixed 6 bushels less of each, he would have put in 6 bushels of barley for every 5 of oats. How many bushels did he mix?

38. A person has a gold watch and a silver one and a chain for both worth \$8. Now, the silver watch and chain are together worth half as much as the gold watch; but when the chain is on the gold watch, they are together worth 3 times as much as the silver watch. What is the value of each?

39 If a certain volume contained 12 more pages, with 3 lines more upon a page, the number of lines would be increased 744; but if it contained 8 pages less, and the lines on a page were not so many by 4, the whole number of lines would be diminished 680.

How many pages are there in the book? and how many lines on a page?

40. Two neighbors, A and B, possess 562 acres of land. If A's farm were 4 times, and B's 3 times, as large as each of them is, they would both together have 1924 acres. How many acres has each?

41. Two men owe more money than they can pay. Says A to B, "Give me $\frac{1}{3}$ of your property, and I shall be able to pay my debts." "If you will give me $\frac{1}{4}$ of yours," replies B, "I shall be able to pay my own." The amount of A's debts is \$1500, and of B's, \$2125. How much property has each in his possession?

42. A trader bought at auction two pipes containing wine. For one he gave 8*s.* a gallon; for the other, 10*s.* 6*d.*; and the whole came to \$160. Having sold 25 gallons from the first pipe, and 16 gallons from the second, he mixed the remainder together, and added $15\frac{3}{4}$ gallons of water. Afterwards, $5\frac{3}{4}$ gallons of the mixture leaked out; and the remainder was worth 8*s.* a gallon. How many gallons did each pipe contain?

43. A man had 32 gallons of wine, in two barrels. Wishing to have an equal quantity in each, he poured out of the first into the second as much as it already contained; again, he poured out of the second into the first as much as it then contained; and, finally, he poured out of the first into the second as much as still remained in it. Each barrel then contained the same quantity. How many gallons did they contain originally?

SECTION VII.

Three or more Unknown Quantities.

1. Three boys, A, B and C, bought fruit at the same time. A bought 4 oranges, 7 peaches and 5 pears, for 51 cents; B bought 6 oranges, 8 peaches and 10 pears, for 74 cents; and C bought 9 oranges, 3 peaches and 2 pears, for 58 cents. What was the price of each?

Let $x =$ the price of an orange,
$y =$ the price of a peach,
and $z =$ the price of a pear.

A. Then $4x + 7y + 5z = 51$, by A's purchase
B. $6x + 8y + 10z = 74$, by B's,
C. $9x + 3y + 2z = 58$, by C's.

These unknown quantities must be made to disappear, one at a time. Either method of elimination, explained in the last section, may be used. *We must, in the first place, deduce, from these three equations, two others, which shall contain but two unknown quantities each.*

If we multiply equation A by 2, the coefficient of z will be the same as it is in equation B.

D. $8x + 14y + 10z = 102$
B. $6x + 8y + 10z = 74$

E. $2x + 6y \quad * \quad = 28$, by subtraction.
F. $x + 3y = 14$, by division.

Again, if we multiply C by 5, the coefficients of z in B and C, will be alike.

$$\begin{aligned} \text{G.}\quad 45\,x + 15\,y + 10\,z &= 290 \\ \text{B.}\quad 6\,x + 8\,y + 10\,z &= 74 \\ \hline \text{H.}\quad 39\,x + 7\,y \quad * \quad &= 216, \text{ by subtraction.} \end{aligned}$$

We have now two equations, namely, F and H, which contain only two unknown quantities.

$$\text{F.}\quad x + 3\,y = 14,$$
$$\text{I.}\quad \text{and } x = 14 - 3\,y, \text{ by transposition.}$$

If we substitute this value of x in equation H, we shall have

$$\text{K.}\quad 39\,(14 - 3\,y) + 7\,y = 216.$$
$$546 - 117\,y + 7\,y = 216, \text{ by multiplication.}$$
$$546 - 216 = 117\,y - 7\,y, \text{ by transposition.}$$
$$330 = 110\,y, \text{ by reduction of terms.}$$
$$\text{L.}\quad y = 3, \text{ by division.}$$

If we substitute this value of y in equation F, we shall have

$$x + (3 \times 3) = 14.$$
$$\text{M.}\quad x = 14 - 9 = 5, \text{ by transposition.}$$

And if we substitute the values of x and y, as determined in L and M, in any of the preceding equations containing z, we shall obtain the value of that quantity. Take equation C, for instance.

$$\text{C.}\quad 9 \times 5 + 3 \times 3 + 2\,z = 58,$$
$$\text{or } 45 + 9 + 2\,z = 58.$$

$2z = 58 - 45 - 9 = 4$, by transposition

and $z = 2$.

ANS. An orange, 5 cents,
A peach, 3 cents;
A pear, 2 cents.

2 A fruiterer sold to A 5 oranges, 6 peaches and 7 pears, for 75 cents; to B 8 oranges, 9 peaches and 5 apples, for 94 cents; to C 2 oranges, 8 pears and 10 apples, for 56 cents; and to D 3 peaches, 6 pears and 9 apples, for 48 cents. What was the price of each?

To solve this question, we must use four unknown quantities; but their values may be found according to the principles already explained. It will be observed, that all the unknown quantities do not enter into each of the equations.

Let $v =$ the price of an orange,
$x =$ the price of a peach,
$y =$ the price of a pear,
and $z =$ the price of an apple.

A. Then $5v + 6x + 7y = 75$, by A's purchase,
B. $8v + 9x + 5z = 94$, by B's,
C. $2v + 8y + 10z = 56$, by C's,
D. $3x + 6y + 9z = 48$, by D's.

C. $2v + 8y + 10z = 56$,
or $2v = 56 - 8y - 10z$, by transposition,
E. and $v = 28 - 4y - 5z$, by division.

By substituting this value of v, in equations A and B we have

$5(28-4y-5z)+6x+7y=75$, from equation A

$140-20y-25z+6x+7y=75$, by multiplication.

F. $6x-13y-25z=-65$, by transposition and addition.

And $8(28-4y-5z)+9x+5z=94$, from equation B

$224-32y-40z+9x+5z=94$, by multiplication.

G. $9x-32y-35z=-130$, by transposition & addition.

We have now three equations, namely, D, F and G, containing but three unknown quantities, x, y and z.

If we multiply equation D by 2, the coefficients of x, in D and F, will be made equal.

H. $6x+12y+18z=\quad 96$

F. $6x-13y-25z=-65$

I. $*\quad 25y+43z=\quad 161$, by subtraction.

Again, if we multiply equation D by 3, the coefficients of x, in D and G, will be the same.

K. $9x+18y+27z=\quad 144$

G. $9x-32y-35z=-130$

L. $*\quad 50y+62z=\quad 274$, by subtraction.

The two equations, I and L, contain but two unknown quantities, which may be eliminated in the usual manner.

If we multiply equation I by 2, we shall have

M. $50y+86z=322$

L. $50y+62z=274$

$*\quad 24z=48$, by subtraction,

and $z=2$, the price of an apple.

By putting this value of z in equation L, we have

$$\text{L.}\quad 50\,y + 124 = 274,$$
$$\text{or } 50\,y = 274 - 124 = 150,$$
$$\text{and } y = 3, \text{ the price of a pear.}$$

By substituting the values of y and z in equation D, we obtain

$$\text{D.}\quad 3\,x + 18 + 18 = 48,$$
$$\text{or } 3\,x = 48 - 36 = 12,$$
$$\text{and } x = 4, \text{ the price of a peach.}$$

And, by putting the values of x and y in equation A, we have

$$5\,v + 24 + 21 = 75,$$
$$\text{or } 5\,v = 75 - 45 = 30,$$
$$\text{and } v = 6, \text{ the price of an orange.}$$

When the value of one of the unknown quantities is determined, it may be substituted for that quantity in either of the equations. Any or all of the different methods of elimination may be used; and it matters not in what order the equations are compared together.

3. A miller sold to one man 12 bushels of wheat, 10 of rye and 16 of barley, for £9 2*s.*; to another, 7 bushels of wheat, 20 of rye and 10 of barley, for £8 19*s.*; and to a third, 15 bushels of wheat, 8 of rye and 20 of barley, for £10 5*s.* What was the price of each per bushel?

4. Divide 125 into four such parts, that, if the first be increased by 4, the second diminished by 4, the third multiplied by 4, and the fourth divided by 4, the sum, difference, product and quotient shall all be equal.

5. Find 3 such numbers, that the first with $\frac{1}{2}$ of the other two, the second with $\frac{1}{3}$ of the other two, and the third with $\frac{1}{4}$ of the other two, shall each be equal to 34.

6. If A and B can perform a piece of work in 8 days, A and C in 9 days, and B and C in 10 days, in how many days can each alone perform the same work?

7. Says A to B and C, "If each of you will give me 10 cents, my money will be, to what you will both have left, as 4 to 5." Says B to A and C, "If each of you will give me 10 cents, my money will be, to what you will then have, as 5 to 4." Says C to A and B, "If you will give me 10 cents each, I shall have twice as much money as both of you." How many cents has each?

8. Three persons divided a sum of money between them in such a manner, that the shares of A and B together amounted to \$900; the shares of A and C, to \$800; and the shares of B and C, to \$700. Required the sum divided, and the share of each

9. A man, with his wife and son, talking of their ages, said, that his age, added to that of his son, was 16 years more than that of his wife; the wife said, that her age added to that of her son, made 8 years more than that of her husband; and that all their ages together amounted to 88 years. What was the age of each?

10. There are two such fractions, that if 3 be added to the numerator of the first, its value is double that of the second; but if 3 be added to the denominator,

their values are equal. Now, the sum of the two fractions is 9 times as great as their difference; and if the numerator of their product be increased by 10, its value will be equal to that of the first fraction. Required the fractions.

11. A grocer has four kinds of tea, marked A, B, C and D. When he mixes together 7 pounds of A, 5 of B and 8 of C, the mixture is worth \$1,21 a pound. When he mixes together 3 pounds of A, 10 of C and 5 of D, the mixture is worth \$1,50 a pound. At one time he sold 8 pounds of A, 10 of B, 10 of C and 7 of D, for \$48; and, at another time, he sold 18 pounds of A and 15 of D, for \$48. What was a pound of each worth?

12. There are two fractions having the same denominator. Now, if 1 be subtracted from the numerator of the smaller, its value will be $\frac{1}{3}$ of the larger fraction; but if 1 be subtracted from the numerator of the larger, its value will be double that of the smaller. And if the numerator of the smaller be subtracted from that of the larger, the value of the fraction will be $\frac{1}{3}$. What are the fractions?

13. Three travellers, A, B and C, have a bill at a tavern, which neither of them alone is able to pay. Says A to B, "If you will lend me $\frac{1}{2}$ of your money, I can pay the bill." B says to C, "I will pay it, if you will lend me $\frac{1}{4}$ of your money." But C concludes to borrow $\frac{1}{8}$ of A's money, and pay the bill himself, which amounts to $6\frac{1}{2}$ dollars. How much money has each of the travellers?

SECTION VIII.

Questions for Practice.

1. A vintner sold to one man 16 dozen of sherry wine and 19 dozen of port, for $382; and to another man, 24 dozen of sherry and 17 dozen of port, for $458;— the prices being the same to both. What was the price of each kind of wine?

2. Two sisters having bought some lace, says Mary to Ann, "Give me a yard of yours, and I shall have as much as you." Ann replied, "If you will give me a yard of yours, I shall have twice as much as you." How many yards had each?

3. Required two such numbers, that, if $\frac{1}{3}$ of the first be added to $\frac{5}{6}$ of the second, the sum shall be 66; but if $\frac{5}{6}$ of the first be added to $\frac{1}{3}$ of the second, their sum shall be 60.

4. Two persons, A and B, talking of their ages, says A to B, "12 years ago I was twice as old as you; and in 12 years my age will be to yours as 3 to 2." What is the age of each?

5. Three young men, A, B and C, speaking of their money, A says to B and C, "If each of you will give me $5, I shall have just half as much as both of you will have left." B says to A and C, "If each of you will give me $5, I shall have just as much as both of you will have left." C says to A and B, "If each of you will give me $5, I shall have twice as much as both of you will have left." How much has each?

6. If you add 2 to the numerator of a certain frac-

tion, its value becomes $\frac{1}{2}$; but if you add 2 to its denominator, the fraction will be equal to $\frac{1}{4}$. What is the fraction?

7. Three factory girls, A, B and C, weave 62 yards of cloth in a given time. A weaves 4 times as many yards as C, added to twice the yards woven by B; and twice A's part, added to 3 times B's, is equal to 17 times C's. How many yards does each weave?

8. Find three such numbers, that $\frac{5}{6}$ of the first, $\frac{2}{5}$ of the second, and $\frac{2}{7}$ of the third, shall be 50; $\frac{2}{3}$ of the first, and twice the difference of the third and second, shall be 40; and 10 less than $\frac{5}{12}$ of the sum of all the numbers shall be 30.

9. Two men, A and B, are employed to set up 220 rods of fence. If A work 9 days and B 8, the fence will not be completed by 2 rods; but if A work 8 days and B 9, they will be able to finish the fence and 4 rods more. How many rods can each build in a day?

10. A farmer employed three laborers, A, B and C, to work at different times. At one time, A and B together earned \$56 in 8 weeks; at another time, A and C earned \$54 in 9 weeks; and, at another, B and C earned \$50 in 10 weeks. What did he pay each man for a week's work?

*11. If you divide the greater of two numbers by the less, the quotient will be 7; and the amount of the numbers is 1008. Required the numbers.

12. A farmer, to pay a debt to a trader, agrees to fill a certain chest with a mixture of corn and oats the corn being 5 shillings a bushel, and the oats 3

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shillings. If he delivers 7 bushels of oats, and the balance of the debt in corn, the chest will not be full by 2 bushels; but if he delivers 6 bushels of corn, and then fills the chest with oats, 6 shillings of the debt will remain unpaid. What is the amount of the debt? How many bushels does the chest hold? How many bushels of each kind of grain must he deliver?

13. A boy bought, at one time, 5 apples, 6 pears and 4 peaches, for 44 cents; at another time, 7 pears, 5 peaches and 3 oranges, for 56 cents; at another, 8 apples, 12 peaches and 5 oranges, for 89 cents; and at another, 10 apples, 3 pears and 9 oranges, for 7[illegible] cents. What did he pay for each kind of fruit?

14. Three persons, A, B and C, talking of their money, A says to B and C, "Give me $\frac{1}{2}$ of your money, and I shall have \$85." B says to A and C, "Give me $\frac{1}{3}$ of your money, and I shall have \$85." C says to A and B, "Give me $\frac{1}{4}$ of your money, and I shall have \$85." What has each?

15. A gentleman gave \$4350 for a house-lot, the land being valued at \$2 a foot. If it had been 6 feet wider, it would have cost \$5394. What were the length and breadth of the lot?

16. I have a certain number of cents in each hand. If I put 10 out of my left hand into my right, there will be twice as many in my right as remain in my left; but if I put 10 out of my right hand into my left, there will be three times as many in my left hand as remain in my right. How many cents have I in each hand?

17. Three boys, A, B and C, play with marbles.

First, A loses to B and C as many as each of them has. Next, B loses to A and C as many as each of them now has. Lastly, C loses to A and B as many as each of them now has. After all, each of them has 16 marbles. How many had each at first?

18. The fore-wheel of a coach makes 5 revolutions while the hind-wheel is making 4; but if the circumference of each were one yard greater, their revolutions would be to each other as 6 to 5. What is the circumference of each in feet?

19. Three sportsmen, before they separated in the fields, agreed to make an equal division of whatever game they might take. During the day, they *bagged* 96 birds; and, in order to divide them equally, A gave B and C as many as they took; next, B gave A and C as many as they then had; and, finally, C shared in the same manner with A and B, when they all had the same number. How many birds were taken by each?

20. A merchant sent ventures to sea for his three children, A, B and C, upon the condition that they should make an equal division of the proceeds. When the ship arrived, A gave to B and C a sum equal to $\frac{1}{3}$ of their ventures. In like manner, B gave to A and C a sum equal to $\frac{1}{3}$ of what they then had. And C found, if he gave to A and B a sum equal to $\frac{1}{3}$ of what they already had, that they would each have 64 dollars. To how much did each of the ventures amount?

CHAPTER IX.

EXERCISES IN GENERALIZATION

WE have already seen that Algebra enables us to solve various questions with facility, and to obtain their true answers in numbers. But this is not its only, nor even its highest office. *It also enables us to deduce general truths from particular instances, and thus to form rules for conducting numerical calculations.* This use of Algebra will be illustrated by a few miscellaneous examples.

1. Add together the sum and the difference of 294 and 175.

$$\begin{array}{lr} \text{Their sum,} & 294 + 175 = 469 \\ \text{Their difference,} & 294 - 175 = \underline{119} \\ & \text{Ans. } 588. \end{array}$$

To *generalize* this question, let a represent the greater quantity, and b, the less. Then $a + b$ will be their sum, and $a - b$, their difference· which we are required to add together.

$$\begin{array}{r} a + b \\ \underline{a - b} \\ \text{Ans. } 2a. \end{array}$$

To express the result of this operation in language: *If we add together the sum and the difference of any two quantities, the amount will be equal to twice the greater quantity.*

2. Add together the sum and the difference of 90 and 65, by this *formula* or rule.

3. Add together the sum and the difference of 136 and 97.

4. Add together the sum and the difference of 375 and 129.

5. Subtract the difference of 324 and 278 from their sum.

Their sum,	$324 + 278 =$	602
Their difference,	$324 - 278 =$	46
	Ans.	$556.$

Indicating the two quantities as before, we must subtract $a - b$ from $a + b$.

$$\begin{array}{r} a + b \\ \underline{a - b} \\ 2\,b. \end{array}$$

That is, *if we subtract the difference of two quantities from their sum, the remainder is equal to twice the smaller quantity.*

6. From the sum of 77 and 25 take their difference.

7. From the sum of 139 and 62 take their difference.

8. From the sum of 827 and 364 take their difference.

9. Multiply the sum of 139 and 87 by their difference.

$$\begin{array}{lrr}
\text{Their sum,} & 139 + 87 = & 226 \\
\text{Their difference,} & 139 - 87 = & 52 \\
\hline
 & & 452 \\
 & & 1130 \\
\hline
 & \text{Their product,} & 11752
\end{array}$$

If we represent the two quantities by the same letters as before, we shall have $a + b$, their sum, to be multiplied by $a - b$, their difference.

$$\begin{array}{lrrr}
 & a & + b & \\
 & a & - b & \\
\hline
 & a^2 & + a\,b & \\
 & & - a\,b & - b^2 \\
\hline
\text{Their product,} & a^2 & * & - b^2.
\end{array}$$

To express this formula in words: *The product of the sum of two numbers, multiplied by their difference, is equal to the difference of their second powers.*

10. Multiply $9 + 7$ by $9 - 7$, according to this rule.

11. Multiply $12 + 8$ by $12 - 8$.

12. Multiply $24 + 15$ by $24 - 15$.

13. Given 9310 and 9298, and required the difference of their squares.

The answer to this question can be far more easily obtained by means of the above formula, than by actually involving each of these numbers, and subtracting the second power of the one from the second power of the other. For, *if we multiply their sum by their difference, the product will be the difference of their second powers.*

The last question is solved by means of the formula, in the following manner:—

$$\begin{array}{rl} 9310 & \\ 9298 & \\ \hline 18608, & \text{the sum of the numbers.} \\ 9310 - 9298 = \quad 12, & \text{their difference.} \\ \hline 223296, & \text{the difference of their squares.} \end{array}$$

This principle should be remembered; as calculations of this sort may be very much abridged by an application of it, when the given numbers are large.

14. Given 23136 and 21947, and required the difference of their second powers.

15. Given 15925 and 14987, and required the difference of their second powers.

16. Given 987264 and 978351, and required the difference of their second powers.

17. Given 999999 and 999993, and required the difference of their second powers.

18. What is the second power of 28? Ans. 784.

Let $a = 20$, and $b = 8$; then $a + b = 28$.

$$\begin{array}{l} a + b \\ a + b \\ \hline a^2 + a\,b \\ \quad\;\; + a\,b + b^2 \\ \hline a^2 + 2\,a\,b + b^2. \end{array}$$

Hence it appears, that *the second power of the sum of two quantities is equal to the sum of their second powers, increased by twice their product.*

19. What is the second power of $a - b$?

$$\begin{array}{l} \phantom{a^2-{}} a - b \\ \underline{\phantom{a^2-{}} a - b} \\ a^2 - a\,b \\ \underline{ - a\,b + b^2} \\ a^2 - 2\,a\,b + b^2. \end{array}$$

That is, *the second power of the difference of two quantities is equal to the sum of their second powers, diminished by twice their product.*

20. An artist sold to A and B two pictures for 573 dollars; and B's picture cost 57 dollars more than A's. What was the price of each?

Let $x =$ the price of A's picture.
Then $x + 57 =$ the price of B's.
And $x + x + 57 = 573$, by the question,
or $2\,x + 57 = 573$;
$2\,x = 573 - 57 = 516$, by transposition,
and $x = 258$.
$x + 57 = 315$.

ANS. A's, \$258; B's, \$315.

In this question, we are required to divide 573 into two such parts, that the greater shall exceed the less by 57. In other words, *having the sum and the difference of two numbers given, we are required to find those numbers.*

Let the sum of the numbers be represented by a, and their difference by b; and let $x =$ the smaller number. Then, as the difference of the numbers is b, we have $x + b =$ the larger number.

$$\text{Therefore, } x + x + b = a,$$
$$\text{or } 2x + b = a,$$
$$\text{and } 2x = a - b;$$
$$x = \frac{a-b}{2}, \text{ the smaller number.}$$

To express this formula in words: *If from the sum of two numbers we subtract their difference, half of the remainder will be equal to the smaller number.*

$$\text{Again, } x = \frac{a-b}{2}, \text{ as above.}$$

Therefore, $x + b = \frac{a-b}{2} + b$, by adding b to each [member.

Or $x + b = \frac{a-b}{2} + \frac{2b}{2} = \frac{a-b+2b}{2}$;

$$x + b = \frac{a+b}{2}, \text{ the greater number.}$$

That is, *if to the sum of two numbers we add their difference, half of the amount will be equal to the greater number.*

21. The sum of two numbers is 175, and their difference 81. What are the numbers?

$$\text{Here, } a = 175, \text{ and } b = 81.$$

$$\frac{a-b}{2} = \frac{175-81}{2} = \frac{94}{2} = 47, \text{ the smaller number.}$$

$$\frac{a+b}{2} = \frac{175+81}{2} = \frac{256}{2} = 128, \text{ the larger number.}$$

22. The salaries of two men, A and B, amount to \$3529 per annum; and A receives \$721 more than B. What is the salary of each?

23. A gentleman paid \$387 for a horse and chaise: and the chaise cost \$75 more than the horse What was the price of each?

24. A man left an estate of \$9134, to be so divided

between his widow and son, that the former should receive $1486 more than the latter. What was the share of each?

25. A's farm contains 12 acres more than B's; and the two farms together contain 162 acres. Required the number of acres in each.

26. Says A to B, "If you will give me 16 dollars, I shall have as much money as you; and we both have 130 dollars." How much money has each?

27. A cistern has two cocks, which together will discharge 203 gallons of water in an hour, and one cock will discharge 13 gallons more than the other. How much will each discharge if running alone?

28. How much tea, at 5*s.* per pound, must be given in barter for 75 gallons of wine at 8*s.* per gallon?

Let $x =$ the pounds of tea.

Then $5\,x = 75 \times 8 = 600$;

and $x = 120$.

ANS. 120 lbs.

In general terms. How much tea, at a per pound, must be given in barter for b gallons of wine at c per gallon?

Let $x =$ the pounds of tea.

Then $a\,x = b\,c$,

and $x = \frac{b\,c}{a}$.

To express this formula for *Barter* in words: *Multiply the given quantity by its price, and divide the product by the price of the quantity required.*

29. How much oil, at 4*s.* a gallon, must I give in barter for 58 barrels of flour at 5 dollars a barrel?

30. How much loaf-sugar, at 25 cents a pound, must be given in barter for 625 yards of cotton, at 13 cents a yard?

31. A man sells 17 acres of land at 30 dollars per acre, and is to receive his pay in flour at 5 dollars a barrel. How many barrels must he receive?

32. How many gallons of wine, at $1,75, must I give in exchange for 1295 pounds of shot, at 5 cents a pound?

33. Two merchants, A and B, engage in a speculation, and gain $750, of which B is to have 4 times as much as A. What is the share of each?

Let $x =$ A's share,
and $4\,x =$ B's share.
Then $x + 4\,x = 750$,
or $5\,x = 750$,
and $x = 150$, A's share.
$4\,x = 600$, B's share.

In this question, a given sum is to be divided between two persons in unequal proportions. It may be generalized as follows:—

Represent the sum to be divided by a.
Let $x =$ A's share,
and $4\,x =$ B's share.
Then $x + 4\,x = a$,
or $5\,x = a$,
and $x = \frac{a}{5}$, A's share.
$4\,x = \frac{4\,a}{5}$, B's share.

Now, whatever sum may be represented by a, it is evident that A is to receive $\frac{1}{5}$ part of it, and B $\frac{4}{5}$ of it.

If a, the sum to be divided, be \$1000, $\frac{a}{5} = 200$, and $\frac{4a}{5} = 800$.

If a represent \$1500, $\frac{a}{5} = 300$, and $\frac{4a}{5} = 1200$.

This question may be still farther generalized, if it be announced as follows:—

Two merchants, A and B, engage in a speculation, and gain a dollars, of which B is to have b times as much as A. What is the share of each?

Let $x =$ A's part of the gain.

Then $b\,x =$ B's part.

And $x + b\,x = a$, by the question,

or $(1 + b)\,x = a$,

and $x = \frac{a}{1+b}$, A's part.

$b\,x = \frac{a}{1+b} \times b = \frac{ab}{1+b}$, B's part.

In the equation $x + b\,x = a$, the first member, $x + b\,x$, is equal to $(1 + b)\,x$; therefore, $(1 + b)$ may be regarded as the coefficient of x.

34. By this formula, divide \$1200 between two men in such a manner, that one shall have 5 times as much as the other.

Here, $a = 1200$, and $b = 5$.

Therefore, $\frac{a}{1+b} = \frac{1200}{1+5} = \frac{1200}{6} = 200$.

And $\frac{ab}{1+b} = \frac{6000}{1+5} = \frac{6000}{6} = 1000$.

Ans. \$200; and \$1000

35. A father says to his son, "I am four times as old as you, and the sum of our ages is 85 years." Required the age of each.

36. Divide 2592 dollars between two men in such a manner, that one shall have 7 times as much as the other.

37. Says A to B, "Our estates together are worth 36465 dollars; but my estate is worth only one fourth part as much as yours." What is the value of each estate?

38. A merchant sold a quantity of blue, black and mixed broadcloths, for \$5000; the black at \$10, the blue at \$11, and the mixed at \$7 per yard. There were twice as many yards of the blue as of the black, and as many yards of the mixed as of both the others. How many yards of each color did he sell?

$$\text{Let } x = \text{the yards of black,}$$
$$2\,x = \text{the yards of blue,}$$
$$\text{and } 3\,x = \text{the yards of mixed.}$$
$$\text{Then } 10\,x + 22\,x + 21\,x = 5000, \text{ by the question,}$$
$$\text{or } 53\,x = 5000.$$
$$\text{And } x = \tfrac{5000}{53} = 94\tfrac{18}{53}, \text{ yards of black.}$$
$$2\,x = 188\tfrac{36}{53}, \text{ yards of blue.}$$
$$3\,x = 283\tfrac{1}{53}, \text{ yards of mixed.}$$

All the answers to the above question contain fractions. I now wish to ascertain what sum I must substitute for \$5000, in order that the answers may consist of whole numbers. For this purpose, I will solve the question by using a instead of the given sum.

If I denote the yards of black by x, the yards of

blue by $2x$, and the yards of mixed by $3x$, as above, I shall have the following equation:—

$$10x + 22x + 21x = a,$$
$$\text{or } 53x = a,$$
$$\text{and } x = \frac{a}{53}.$$

Any number, divisible by 53 without a remainder,—that is, the product of 53 multiplied by any number whatever,—may be put in the place of a, and the answer will be free from fractions.

Suppose, for instance, the value of the whole quantity sold to be 2491 dollars, which is the product of 53 multiplied by 47. Then he would have sold 47 yards of black, 94 yards of blue, and 141 yards of mixed.

39. In a certain school, $\frac{1}{4}$ of the pupils learn navigation, $\frac{1}{5}$ learn geometry, $\frac{1}{6}$ learn algebra, and all the rest, a, learn arithmetic. How many pupils are there in all?

$$\text{Let } x = \text{the number of pupils.}$$
$$\text{Then } x = \frac{x}{4} + \frac{x}{5} + \frac{x}{6} + a,$$
$$\text{or } 60x = 15x + 12x + 10x + 60a;$$
$$60x = 37x + 60a, \text{ by addition};$$
$$23x = 60a, \text{ by transposition};$$
$$\text{and } x = \frac{60a}{23}.$$

Instead of a, in the question, use any number that can be divided by 23 without a remainder, and the answer will be a whole number.

Substitute, for instance, 46 in the place of a.

Ans. 120 pupils.

The last two examples are intended to exhibit the

manner in which questions may be prepared, whose answers shall be free from fractions.

40. Three men, A, B and C, trade in company, and gain \$1350. Now, if A put into the joint stock \$7 as often as B put in \$6, and B put in \$6 as often as C put in \$5, what is each man's share of the gain?

$$\text{Let } x = \text{A's share.}$$

$$\text{Then } \frac{6x}{7} = \text{B's share,}$$

$$\text{and } \frac{5x}{7} = \text{C's share.}$$

Therefore, $x + \frac{6x}{7} + \frac{5x}{7} = 1350$, by the question,

or $7x + 6x + 5x = 9450$, by multiplication.

$18x = 9450$, by adding terms,

and $x = 525$.

$$\text{Therefore, } \frac{6x}{7} = 450,$$

$$\text{and } \frac{5x}{7} = 375.$$

Ans. A, \$525; B, \$450; C, \$375

To generalize this question, we will suppose A put in a dollars as often as B put in b dollars; and that B put in b dollars as often as C put in c dollars; that is, we will use the letters a, b, c, instead of the numbers 7, 6, 5. Let the amount gained be represented by g

$$\text{Let } x = \text{A's share of the gain.}$$

$$\text{Then } \frac{bx}{a} = \text{B's share,}$$

$$\text{and } \frac{cx}{a} = \text{C's share.}$$

Then $x + \frac{bx}{a} + \frac{cx}{a} = g$, by the question.

$$ax + bx + cx = ag, \text{ by multiplication}$$
$$\text{or } (a + b + c)\, x = ag,$$
$$\text{and } x = \frac{ag}{a+b+c}, \text{ A's share.}$$
$$\text{Then } \frac{bx}{a}, \text{ or } \frac{b}{a}\,x = \frac{bg}{a+b+c}, \text{ B's share,}$$
$$\text{and } \frac{cx}{a}, \text{ or } \frac{c}{a}\,x = \frac{cg}{a+b+c}, \text{ C's share.}$$

That is, *To find each man's share of the gain, we must multiply the whole gain by his share of the stock, and divide the product by the whole stock.*

If there has been a *loss*, instead of a *gain*, each man's share of the loss can be ascertained in a similar manner. This is a general rule for *Fellowship*, when all the stock has been employed the same time.

41. Divide 8736 dollars among three men in such a manner, that their shares shall be to each other as the numbers 3, 4 and 5, respectively.

42. Two traders, A and B, found that they had gained, at the end of the year, 3792 dollars. A having put \$5000 and B \$7000 into the joint stock, what is each man's share of the gain?

43. Three merchants, A, B and C, made up a joint voyage, by which they lost 2595 dollars. A furnished \$300 of the capital, as often as B furnished \$500, and C, \$700. What is the loss of each?

44. The premium on a policy of insurance is \$513, which is to be divided between three underwriters. A took 3 shares of the risk, B 7 shares, and C 9 shares. What part of the premium belongs to each?

45. Three merchants, A, B and C, traded in company, and gained \$8840. They severally contributed

to the joint stock, in the proportion of 5, 6 and 7; and A's money was used 4 months; B's, 3 months, and C's, 2 months. What part of the gain belongs to each?

$$\text{Let } x = \text{A's share of the stock.}$$

$$\text{Then } \frac{6x}{5} = \text{B's share,}$$

$$\text{and } \frac{7x}{5} = \text{C's share.}$$

Now, as A furnished x dollars for 4 months, he furnished what was equal to $4x$ dollars for one month. And B's $\frac{6x}{5}$ dollars for 3 months $= \frac{18x}{5}$ dollars for one month; and C's $\frac{7x}{5}$ dollars for 2 months $= \frac{14x}{5}$ dollars for one month.

$$\text{Therefore, } 4x = \text{A's share of the gain,}$$

$$\frac{18x}{5} = \text{B's share,}$$

$$\text{and } \frac{14x}{5} = \text{C's share.}$$

$$\text{Then } 4x + \frac{18x}{5} + \frac{14x}{5} = 8840, \text{ by the question}$$

$$20x + 18x + 14x = 44200,$$

$$\text{or } 52x = 44200,$$

$$\text{and } x = 850.$$

$$\text{Then } 4x = 3400, \text{ A's share of the gain,}$$

$$\frac{18x}{5} = 3060, \text{ B's share,}$$

$$\text{and } \frac{14x}{5} = 2380, \text{ C's share.}$$

Let us now suppose that the sum gained was s dollars; that A put in a dollars; B, b dollars; and C, c

dollars; and that A's money was in h months; B's, m months; and C's, n months.

$$\text{Let } x = \text{A's share of the stock.}$$

$$\text{Then } \frac{bx}{a} = \text{B's share,}$$

$$\text{and } \frac{cx}{a} = \text{C's share.}$$

Now, if we multiply each man's stock by its time, as before, we shall have

$$h\,x = \text{A's share of the gain,}$$

$$\frac{bmx}{a} = \text{B's share,}$$

$$\text{and } \frac{cnx}{a} = \text{C's share.}$$

$$\text{Then } h\,x + \frac{bmx}{a} + \frac{cnx}{a} = s, \text{ by the question.}$$

$$a\,h\,x + b\,m\,x + c\,n\,x = a\,s, \text{ by multiplication,}$$

$$\text{or } (a\,h + b\,m + c\,n)\,x = a\,s,$$

$$\text{and } x = \frac{as}{ah+bm+cn}.$$

$$\text{Therefore, } h\,x = \frac{ahs}{ah+bm+cn}, \text{ A's share,}$$

$$\frac{bmx}{a} = \frac{bms}{ah+bm+cn}, \text{ B's share,}$$

$$\text{and } \frac{cnx}{a} = \frac{cns}{ah+bm+cn}, \text{ C's share.}$$

Having found the value of x, we obtain the formulas for the shares of A, B and C, by substitution. For instance, A's share of the gain being $h\,x$, we multiply the value of x by h, and obtain $\frac{ahs}{ah+bm+cn}$; and so with the others.

Hence, to find the share of either of the copart-

ners, *Multiply the sum to be divided by the product of his share of the stock and the time it was used, for a dividend.*

Multiply the several shares by their respective times and divide the dividend by the sum of their products.

46. Three men, A, B and C, traded in company, and gained 2160 dollars. A put in \$400 for 5 months; B, \$500 for 2 months; and C, \$600 for 4 months. What is each man's share of the gain?

47. Three men, A, B and C, hired a pasture in common, for 82 dollars. A put in 3 horses for 4 months; B, 4 horses for 2 months; and C, 7 horses for three months. How much must each man pay?

48. Two merchants, A and B, engaged in a speculation, by which they gained 2100 dollars. The capital employed was \$14000, of which A furnished \$5000, and B, \$9000; but A's money was used a year, whereas B's was used only 5 months. Required the gain of each.

49. A farmer would mix corn at a shillings per bushel with oats at b shillings per bushel, so that a bushel of the mixture may be worth c shillings. How many bushels of each sort must he take, to make a composition of n bushels?

Let $x =$ the bushels of corn.
Then $n - x =$ the bushels of oats.
Also, $a x =$ the value of the corn,
$b n - b x =$ the value of the oats,
and $c n =$ the value of the mixture.
Then $a x + b n - b x = c n$, by the question.

$$a x - b x = c n - b n, \text{ by transposition,}$$
$$\text{or } (a - b) x = c n - b n,$$
$$\text{and } x = \frac{cn - bn}{a - b}, \text{ the bushels of corn.}$$
$$\text{Also } n - x = n - \frac{cn - bn}{a - b}, \text{ or } \frac{an - cn}{a - b}, \text{ the oats.}$$

50. A grocer would mix sugar at 10*d.* per pound, with another sort, worth 7*d.* per pound, so as to make 1 cwt. worth 8*d.* per pound. How many pounds of each sort must he take?

51. A dealer in oils would furnish 36 gallons at 4 shillings a gallon; and he has but two kinds on hand, one of which is worth a dollar, and the other, 3 shillings, a gallon. How many gallons of each must he take.

52. Two persons, A and B, set out from one place, and both go the same road; but A goes a hours before B, and travels n miles an hour; B follows, and travels m miles an hour. If m be greater than n, in how many hours will B overtake A? and how many miles will he travel?

$$\text{Let } x = \text{the hours B travelled.}$$
$$\text{Then } x + a = \text{the hours A travelled.}$$
$$\text{Also, } m x = \text{the miles B travelled,}$$
$$\text{and } n x + n a = \text{the miles A travelled.}$$
$$\text{Therefore, } m x = n x + n a,$$
$$\text{or } m x - n x = n a,$$
$$\text{and } x = \frac{n a}{m - n}, \text{ the hours B travelled.}$$
$$x + a = \frac{n a}{m - n} + a, \text{ or } \frac{m a}{m - n}, \text{ the hours A travelled}$$
$$m x = \frac{n a}{m - n} \times m, \text{ or } \frac{m n a}{m - n}, \text{ the miles travelled.}$$

53. If A sets out 10 hours before B, and travels 3 miles an hour, and B travels 4 miles an hour, how long and how far will they travel before they come together?

54. Suppose that A sets out 6 hours before B, and travels 3 miles an hour, and that B travels 5 miles an hour; how long and how far will they travel?

55. Divide a line c feet long into two parts, so that one part may be to the other as m to n.

$$\text{Let } x = \text{ one of the parts.}$$

$$\text{Then } \frac{m x}{n} = \text{ the other part.}$$

$$\text{Therefore, } x + \frac{m x}{n} = c, \text{ by the question,}$$

$$\text{or } n x + m x = c n,$$

$$\text{and } x = \frac{c n}{n + m}, \text{ one of the parts.}$$

$$\text{Also } \frac{m x}{n} = \frac{c m}{n + m}, \text{ the other part.}$$

56. Let the line be 20 feet long, and be so divided that one part shall be to the other as 3 to 4.

57. Divide a line 36 inches long into two parts, which shall be to each other as 4 to 5.

58. Divide any number, a, into two such parts, that their product shall be to the square of the greater part, as m to n.

$$\text{Let } x = \text{ the greater part.}$$

$$\text{Then } a - x = \text{ the smaller part.}$$

$$\text{Also } a x - x^2 = \text{ their product,}$$

$$\text{and } x^2 = \text{ the square of the greater part.}$$

$$\text{Therefore, } x^2 = \frac{a n x - n x^2}{m},$$

$$\text{or } m x^2 = a n x - n x^2,$$

$$\text{and } m x = a n - n x, \text{ by dividing by } x.$$

$$m x + n x = a n, \text{ by transposition,}$$

$$\text{and } x = \frac{a n}{m+n}, \text{ the greater part.}$$

$$a - x = a - \frac{a n}{m+n}, \text{ or } \frac{a m}{m+n}, \text{ the less part.}$$

59. Let the number to be divided by these formulas be 100, and the ratio given, as 2 to 3.

60. Divide 100 into two such parts, that their product shall be $\frac{1}{3}$ the square of the greater part.

61. What number is that to which if we add a, subtract b from the sum, multiply the remainder by c, and divide the product by d, the quotient shall be equal to the number? Ans. $\frac{ac-bc}{d-c}$.

62. What number is that, to which if we add 40, subtract 25 from the sum, multiply the remainder by 6, and divide the product by 9, the quotient and the number shall be equal?

63. Required the number, to which if we add 84, subtract 38 from the sum, multiply the remainder by 5, and divide the product by 7, the quotient shall be equal to the number sought.

64. A trader, having gained \$4051 by his business, and lost \$3186 by bad debts, found that $\frac{5}{8}$ of what he had left equalled the capital with which he commenced trade. What was his capital?

65. A farmer paid a men and b boys m dollars for

working a day; he afterwards paid c men and d boys n dollars for working a day. What was the pay of each?

$$\text{Let } x = \text{the wages of a man,}$$
$$\text{and } y = \text{the wages of a boy.}$$
$$\text{A. Then } a\,x + b\,y = m,$$
$$\text{B. and } c\,x + d\,y = n.$$

Multiply equation A by c, and equation B by a, to make the coefficients of x equal; and we have

$$\text{C. } a\,c\,x + b\,c\,y = c\,m$$
$$\text{D. } a\,c\,x + a\,d\,y = a\,n$$
$$b\,c\,y - a\,d\,y = c\,m - a\,n, \text{ by subtraction,}$$
$$\text{or } (b\,c - a\,d)\,y = c\,m - a\,n,$$
$$\text{and } y = \frac{c\,m - a\,n}{b\,c - a\,d}, \text{ a boy's wages.}$$

By substituting this value of y in equation A, we obtain

$$x = \frac{b\,n - d\,m}{b\,c - a\,d}, \text{ a man's wages.}$$

66. A farmer paid 5 men and 8 boys \$9 for working a day; he afterwards paid 7 men and 6 boys \$10 for working a day. What were the wages of a man and a boy?

67. A man lent a certain sum of money; and, at the end of 4 years, he received, for principal and interest, \$775. What was the sum lent, the interest being reckoned at 6 per cent.?

$$\text{Let } x = \text{the principal.}$$

Then, since 6 per cent. implies \$6 for the use of \$100, or $\frac{6}{100}$ of the principal, as the annual interest,

$$\frac{6x}{100} = \text{the interest for 1 year,}$$

$$\text{and } \frac{24x}{100} = \text{the interest for 4 years.}$$

$$\text{Of course, } x + \frac{24x}{100} = \text{the amount.}$$

$$\text{Therefore, } x + \frac{24x}{100} = 775, \text{ by the question.}$$

$$100x + 24x = 77500, \text{ by multiplication,}$$

$$\text{or } 124x = 77500,$$

$$\text{and } x = 625.$$

ANS. \$625

68. To generalize this question, we have the amount, rate and time given, to find the principal.

In solving this problem, and others of a similar sort, we shall find it convenient to indicate the several quantities, given and required, by their initials, instead of the letters commonly used; although this is not essential.

Let p = the *principal*, or sum lent.
r = the annual *rate* per cent.
t = the *time*.
i = the *interest*.
a = the *amount*, which always $= p + i$.

In the question above, the value of p is required.

$$r\,p = \text{the interest for one year.}$$

$$t\,r\,p = \text{the interest for } t \text{ years.}$$

$$p + t\,r\,p = \text{the amount.}$$

$$\text{Therefore, } p + t\,r\,p = a,$$

$$\text{or } (1 + t\,r)\,p = a,$$

$$\text{and } p = \frac{a}{1 + t\,r}.$$

To express this formula in words: *Add* 1 *to the product of the time multiplied by the rate, and divide the amount by the sum. The quotient will be the principal.*

This is a general rule for calculating *Discount.* When a man pays a sum of money before it has become due, he is evidently entitled to some reduction from the debt. Equity requires that he should pay such a sum as would amount to the sum due, if put at interest during the time for which it is advanced, at any rate per cent. agreed upon by the parties. The difference between such a sum, which is called the *present worth* of the debt, and the debt itself, is the *discount.*

69. What is the present worth of 392 dollars, due in 2 years, the discount being reckoned at 6 per cent.?

It is here required to find what sum will amount to 392 dollars, in 2 years, at 6 per cent. In other words, the amount, time and rate are given, to find the principal.

70. A gentleman hired a sum of money at 5 per cent.; and at the end of 3 years, he paid 8234 dollars, for principal and interest. What was the sum hired?

71. A merchant sold an invoice of goods, amounting to 1961 dollars, on a year's credit. What discount should he make for present payment, allowing money to be worth 6 per cent.?

72. Required the present worth of 713 dollars, discounted for 4 years at 6 per cent.

73. What sum will amount to \$667 in 3 years, at 5 per cent.?

74. Given the amount, time and principal, to find the rate.

$$r\,p = \text{the interest for 1 year.}$$
$$t\,r\,p = \text{the interest for } t \text{ years.}$$
$$p + t\,r\,p = \text{the amount.}$$
$$\text{Then } p + t\,r\,p = a,$$
$$\text{or } t\,r\,p = a - p,$$
$$\text{and } r = \frac{a-p}{tp}.$$

That is, *From the amount subtract the principal; and divide the remainder by the product of the principal multiplied by the time. The quotient will be the rate.*

75. A man lent \$420; and, in 5 years, he received in payment \$546. At what rate per cent. was the money lent?

76. At what rate per cent. will \$380 amount to \$513, in 7 years?

77. Given the interest, time and rate, to find the principal.

$$r\,p = \text{the interest for 1 year.}$$
$$t\,r\,p = \text{the interest for } t \text{ years.}$$
$$\text{Therefore, } t\,r\,p = i,$$
$$\text{and } p = \frac{i}{t\,r}.$$

Expressed in words: *Multiply the time by the rate, and divide the interest by the product. The quotient will be the principal.*

78. A paid B \$126 for the use of a certain sum of

money 3 years; the interest being reckoned at 5 per cent. What was the sum lent?

79. In the course of 4 years, a man paid interest to the amount of $288, which was reckoned at 6 per cent. What was the debt?

80. The amount, principal and rate being given, to find the time.

$$r\,p = \text{the interest for 1 year.}$$
$$t\,r\,p = \text{the interest for } t \text{ years}$$
$$p + t\,r\,p = \text{the amount.}$$
$$\text{Therefore, } p + t\,r\,p = a,$$
$$\text{or } t\,r\,p = a - p$$
$$\text{and } t = \frac{a-p}{rp}.$$

That is, *From the amount subtract the principal; and divide the remainder by the product of the rate and principal. The quotient will be the time.*

81. A man lent $460 at 5 per cent. and received, for principal and interest, $529. How long was the money kept?

82. In what time will $780 amount to $1014, interest being reckoned at 6 per cent.?

These examples show the manner in which general results, or formulas, are obtained; and also how they may be used in solving particular questions. Let the learner now turn back to Chapter VIII., and generalize the questions marked with a star (*); and then solve the same questions, numerically, by their respective formulas. He will thus be prepared to generalize some of the more difficult questions, in the same chapter, which are not marked.

CHAPTER X.

EVOLUTION.

SECTION I.

Introduction.

When a quantity is multiplied by itself one or more times, the product is called a *Power* of that quantity. Thus, a^2, being the product of $a \times a$, is the second power or square of a; and b^3, that is, $b \times b \times b$, is the third power or cube of b. [See Chap. VII. Sec. I.]

On the contrary, *the quantity which is multiplied by itself to produce any power, is said to be the Root of that power.* Thus, a is the second or square root of a^2; and b is the third or cube root of b^3.

Powers and Roots are, therefore, correlative terms; and Evolution and Involution are the reverse of each other. Involution is the method of raising a given root to a proposed power; but *Evolution is the method of finding the roots of given powers.*

Involution is more perfect, however, than Evolution; for if any proposed power of a given quantity be required, it can be exactly obtained; but there are many quantities whose exact roots cannot be found

It is evident, for instance, that the square root of a cannot be determined; for there is no quantity, which, being multiplied by itself, will produce a.

The roots and powers of numbers have the same relation to each other as those of literal quantities. Thus, the second powers of 2 and 3 are 4 and 9; and the square roots of 4 and 9 are 2 and 3. The exact roots of the intermediate numbers, 5, 6, 7 and 8, cannot be found.

TABLE OF ROOTS AND POWERS.

Roots.	1	2	3	4	5	6	7	8	9	10
Squares.	1	4	9	16	25	36	49	64	81	100
Cubes.	1	8	27	64	125	216	343	512	729	1000
4th powers.	1	16	81	256	625	1296	2401	4096	6561	10000
5th powers.	1	32	243	1024	3125	7776	16807	32768	59049	100000

The roots of quantities are indicated either by means of the *radical sign* $\sqrt{}$, or by a fractional index

Thus, $\sqrt[2]{a}$, or $\sqrt{a}$, is the square root of a.

$\sqrt[3]{a}$ is the cube root of a.

$\sqrt[4]{a^3}$ is the 4th root of a^3.

$\sqrt{64}$ is the square root of 64, which is 8.

$\sqrt[3]{a+x}$ is the cube root of $a+x$.

If the quantity affected by the radical sign be not a complete power, that is, if its root cannot be exactly found, it is called a *Surd*, or *Irrational Quantity*. Thus, $\sqrt{35}$, $\sqrt[3]{x^2}$, $\sqrt[5]{a^3}$, &c., are surd quantities.

Express the roots of the following quantities by means of the radical sign:

1. The square root of x.
2. The fourth root of b^3
3. The cube root of $x^2 + y$.
4. The fifth root of 79.
5. The square root of $a^2 - b + 14$.

When the root of a quantity is expressed by means of a fractional index, *the numerator of the fraction indicates the power of the quantity, and the denominator the root required.*

Thus, $a^{\frac{1}{2}}$ is the square root of a^1 or a.

$a^{\frac{3}{2}}$ is the square root of a^3.

$a^{\frac{1}{3}}$ is the cube root of a.

$(a^2 + b)^{\frac{1}{4}}$ is the fourth root of $a^2 + b$.

$a^{\frac{2}{3}}$ is the cube root of a^2.

The expression $a^{\frac{2}{3}}$ may be regarded either as the second power of the third root of a, or as the third root of the second power of a. And so with all other quantities having fractional indices.

Suppose the value of a to be 27. The third root of 27 is 3, and the second power of 3 is 9. Again, the second power of 27 is 729, and the third root of 729 is 9. The value is the same, whichever mode of expression is used.

Express the roots of the following quantities by means of fractional indices:

6. The square root of x.
7. The fourth root of y^3.
8. The cube root of $(a^2 + x)^2$.

9. The m th root of c.
10. The square root of $x^3 - c^2 + 12$.

If the numerator and denominator of a fractional index be the same, the value of the quantity is not affected by it; for $a^{\frac{2}{2}}$, that is, the second root of the second power of a, is evidently a.

As the value of a fraction is not altered, when both the numerator and denominator are either multiplied or divided by the same number, fractional indices may be changed into other indices of the same value; as, $a^{\frac{1}{2}}$, $a^{\frac{2}{4}}$, $a^{\frac{3}{6}}$, $a^{\frac{4}{8}}$, &c., which are all equal.

Suppose the value of a to be 16. Then the second root of a is 4, whose first power is also 4. Again, the fourth root of a, or 16, is 2; and the second power of 2 is 4. And so with the others.

We can, therefore, reduce different fractional indices to other indices which shall express the same root, by reducing the fractions to a common denominator.

When a letter or figure is prefixed to a quantity affected by the radical sign, it is to be regarded as a coefficient, and the two quantities are supposed to be multiplied together.

Thus, $a\sqrt{x}$ implies that the square root of x is multiplied by a; and $5\sqrt{a^3}$ is the product of the square root of a^3, multiplied by 5. But $5+\sqrt{a^3}$, or $5-\sqrt{a^3}$, implies that the square root of a^3 is to be added to, or subtracted from, 5, and not multiplied by that number.

SECTION II.

Roots of Simple Quantities

1. What is the square root of a^6? Ans. a^3.

We are here required to find two equal factors, whose product shall be a^6; and, as we multiply powers by adding their exponents, [See Chap. VII. Sec. VI.] $a^3 \times a^3 = a^6$. Or the required root may be expressed by a fractional index, thus, $a^{\frac{6}{2}}$; which, the fraction being reduced, becomes a^3.

2. What is the cube root of a^6? Ans. a^2.

Here we are required to find three equal factors, whose continued product shall be a^6; and, by the rule for multiplying powers, $a^2 \times a^2 \times a^2 = a^6$. If the required root be expressed by a fractional index, it will be $a^{\frac{6}{3}} = a^2$, as above.

3. What is the square root of $16\,a^2$? Ans. $4\,a$.

For $4\,a \times 4\,a = 16\,a^2$. The root of the coefficient is found and prefixed to the root of the literal quantity, which is obtained as above.

4. What is the square root of $9\,a^4\,b^2\,x^6$?

Ans. $3\,a^2\,b\,x^3$.

For $3\,a^2\,b\,x^3 \times 3\,a^2\,b\,x^3 = 9\,a^4\,b^2\,x^6$. We divide the exponent of every letter by the index of the required root, and annex the result to the root of the coefficient.

5. What is the cube root of a^5? Ans. $a^{\frac{5}{3}}$.

As the exponent of the given power cannot be divided by the index of the required root, without leav-

ing a remainder, the root must be *represented* by a fractional index.

6. What is the fourth root of $81\,a^4\,c^3$?

Ans. $3\,a\,c^{\frac{3}{4}}$.

7. Required the fifth root of $1024\,a^5\,x^{10}$.

Ans. $4\,a\,x^2$.

From these examples and observations we derive the following RULE for extracting the root of a simple quantity, viz:

Divide the exponent of the given power by the index of the root to be found, and annex the result to the root of the coefficient.

8. What is the square root of $64\,a^4\,b^2$?
9. What is the cube root of $27\,a^3\,b^6\,x^9$?
10. What is the fourth root of $81\,a^8\,x^4\,y^{12}$?
11. Required the fifth root of $32\,x^5\,y^{10}$.
12. Extract the cube root of $64\,a^6\,x^3\,y^{12}$.
13. Required the square root of $5\,a^2\,x^4$.
14. What is the cube root of $7\,x^6\,y^9$?
15. Extract the fourth root of $1296\,a^4\,b^8\,x^{16}$.
16. Required the fourth root of $16\,a^8\,b^3$.
17. What is the third root of $9\,a^3\,b^4\,x^6$?
18. Find the square root of $25\,a^4\,b$.
19. Extract the cube root of $64\,x^4\,y^5$.

The root of a fraction is found in the same manner. *Extract the root of the numerator for a new numerator, and the root of the denominator for a new denominator.*

20. What is the square root of $\frac{9\,a^2}{16\,b^4}$?

21. Required the cube root of $\frac{27\,a^6}{8\,b^6\,c^3}$.

22. What is the fourth root of $\frac{81a^4}{b^8}$?

23. Extract the cube root of $\frac{a^3x^6}{64b^3c^6}$.

24. Required the square root of $\frac{16a^4b^2}{5x^2y^4}$.

25. What is the cube root of $\frac{8c^4x^3}{9a^6b}$?

26. What is the square root of $\frac{2b}{4x^2y^3}$?

To determine what *sign* should be prefixed to a root, observe, in general, that the root, when multiplied by itself the requisite number of times, must re-produce the given power. Therefore,

An ODD *root of any quantity must have the same sign which the quantity has.* The cube root of $-a^3$ is $-a$; for $-a \times -a = +a^2$, and $+a^2 \times -a = -a^3$. And the cube root of $+a^3$ is $+a$; for $+a \times +a = +a^2$, and $+a^2 \times +a = +a^3$.

An EVEN *root of a positive quantity has two signs, the one positive, the other negative. Such a quantity is said to be ambiguous.* The square root of a^2 may be either $+a$, or $-a$; for $+a \times +a = +a^2$, and $-a \times -a = +a^2$, also. When it is not known, from the nature of the question, whether the root is positive or negative, it should be marked with the *ambiguous sign*; thus $\pm a$.

There is no such thing as the EVEN *root of a negative quantity*; for neither $-a \times -a$, nor $+a \times +a$, will produce $-a^2$.

27. What is the square root of $25a^2b^4$?

28. What is the cube root of $125a^3b^6$?

29. What is the square root of $7\,x^4\,y^3$?
30. Required the fifth root of $-243\,x^{10}$.
31. Extract the fourth root of $256\,a^8\,b^4\,c^{12}$.
32. Required the square root of $64\,a^2\,x^4$.
33. Extract the cube root of $-125\,m^6\,n^3$.
34. What is the sixth root of $4096\,x^6\,z^{12}$?
35. Required the cube root of $64\,x^3\,z^6$.
36. Extract the square root of $17\,x^4\,y^3\,z^2$.
37. What is the cube root of $-18\,a^6\,y^4\,z^3$?
38. Required the fifth root of $\frac{a^{10}\,b^5}{18\,c^5\,x}$.
39. Find the cube root of $\frac{12\,a^3\,x^7}{125\,y^6\,x^4}$.

SECTION III.

To extract the Square Root of a Compound Quantity.

1. What is the square root of $12\,x^2\,y^4 + 4\,y^6 + 9\,x^4\,y^2$?

Since the power given in this question consists of three terms, it is evident that its root must contain more than one term; for the second power of a simple quantity is a simple quantity, and the second power of a binomial consists of three terms. [See Chap. VII. Sec. III.] We are, therefore, required to find the binomial, whose square is the quantity proposed in the question.

We may here remark that no binomial, as $x^2 + y^2$, can be a complete power; and the root of an incomplete power can be found only by approximation.

The second power of the binomial $a + b$, is $a^2 + 2\,a\,b + b^2$, [See Chap. IX. Quest. 18.] in which, it will be observed, the several terms are arranged according to the powers of the letter a.

If we arrange the terms of the proposed quantity according to the powers of x, it will become

$$9\,x^4\,y^2 + 12\,x^2\,y^4 + 4\,y^6.$$

In the formula, $(a + b)^2 = a^2 + 2\,a\,b + b^2$, the first term of the power, a^2, is the square of the first term of the root, a.

Hence we infer that the first term of the given quantity, $9\,x^4\,y^2$, is the square of the first term of the root sought, which is, therefore, $3\,x^2\,y$; for $3\,x^2\,y \times 3\,x^2\,y = 9\,x^4\,y^2$.

Again, the second term of the power expressed in the formula, $2\,a\,b$, is twice the product of the two terms of the binomial, $a + b$; of course, if it be divided by twice the first term, a, the quotient is the second term, b.

To apply this principle, we must divide the second term of the given power, $12\,x^2\,y^4$, by twice the first term of the root already obtained, $3\,x^2\,y$; that is, by $6\,x^2\,y$. The quotient is $2\,y^3$, which is the other term of the root required.

Therefore, $3\,x^2\,y + 2\,y^3$ is the whole root sought; and $(3\,x^2\,y + 2\,y^3) \times (3\,x^2\,y + 2\,y^3) = 9\,x^4\,y^2 + 12\,x^2\,y^4 + 4\,y^6$.

The several steps taken in this investigation may be expressed together in the following manner:—

$$
\begin{array}{r|l}
 & 9\,x^4\,y^2 + 12\,x^2\,y^4 + 4\,y^6\ (\,3\,x^2\,y + 2\,y^3 \\
 & 9\,x^4\,y^2 \\
\hline
6\,x^2\,y + 2\,y^3 & 12\,x^2\,y^4 + 4\,y^6 \\
 & 12\,x^2\,y^4 + 4\,y^6 \\
\hline
 & \quad * \qquad\quad *
\end{array}
$$

The given quantity being arranged according to the different powers of x, as in Division, the root of the first term is found to be $3\,x^2\,y$, which is put in the quotient's place.

The second power of $3\,x^2\,y$ is next subtracted from the first term of the given quantity, $9\,x^4\,y^2$, when there is no remainder; which proves that the true root of this term has been found.

To ascertain the next term of the root, we must divide the second term of the given power by twice the root already found; therefore, the remaining terms of the power, $12\,x^2\,y^4 + 4\,y^6$, are brought down for a new dividend.

That portion of the root which has been found, $3\,x^2\,y$, is then doubled, and put in the divisor's place.

The first term of the new dividend, $12\,x^2\,y^4$, is divided by $6\,x^2\,y$, and the quotient is found to be $2\,y^3$, which is the second term of the root.

To ascertain whether $3\,x^2\,y + 2\,y^3$ be the true root required, we should involve it, and subtract its second power from the given quantity, when there must be no remainder. But the second power of the first term, $3\,x^2\,y$, has already been subtracted. Now, in the formula, $(a + b)^2 = a^2 + 2\,a\,b + b^2$, twice the first term of the root, multiplied by the last term,

is equal to the second term of the power; also the last term of the root, multiplied by itself, gives the last term of the power. That is, if we add the last term of the root to twice the first term, and multiply their sum by the last term, the product will be equal to the second and third terms of the power. Therefore, $2\ y^3$ is added to $6\ x^2\ y$, to complete the divisor.

The whole divisor, $6\ x^2\ y + 2\ y^3$, is then multiplied by $2\ y^3$; and the product is subtracted from the new dividend. There being no remainder, we are certain that $3\ x^2\ y + 2\ y^3$ is the true root required.

From this investigation may be derived the following RULE for extracting the square root of a compound quantity.

1. *Arrange all the terms of the given quantity according to the powers of one of the letters, so that the highest power shall stand first, the next highest next, and the rest in order, as in Division.*

2. *Find the root of the first term, and put it in the place of a quotient.*

3. *Subtract the square of this root from the first term of the given quantity, and bring down the remaining terms for a dividend.*

4. *Double the root already found, and put it in the divisor's place.*

5. *See how often this divisor is contained in the first term of the new dividend, and annex the quotient both to the root already found and to the divisor.*

6. *Multiply the divisor, thus increased, by the term of the root last found, and subtract the product from the dividend.*

7. *Double the whole root for a new divisor, and divide as before.*

8. *Proceed in this way, until the entire root of the given quantity is extracted.*

2. What is the square root of $9x^4 - 12x^3 + 16x^2 - 8x + 4$?

$$
\begin{array}{r|l}
 & 9x^4 - 12x^3 + 16x^2 - 8x + 4\,(3x^2 - 2x + 2. \\
 & 9x^4 \\
\hline
6x^2 - 2x & -12x^3 + 16x^2 \\
 & -12x^3 + 4x^2 \\
\hline
6x^2 - 4x + 2 & 12x^2 - 8x + 4 \\
 & 12x^2 - 8x + 4 \\
\hline
 & * \quad * \quad *
\end{array}
$$

In this example, the second term of the root, $2x$, is —, because it is the quotient of $-12x^3$ divided by $+6x^2$. The second divisor, $6x^2 - 4x$, is obtained, as before, by doubling the *whole* of the root already found.

3. What is the square root of $a^2 + 2ab + b^2 - 2ac - 2bc + c^2$?

$$
\begin{array}{r|l}
 & a^2 + 2ab + b^2 - 2ac - 2bc + c^2\,(a + b - c. \\
 & a^2 \\
\hline
2a + b & 2ab + b^2 \\
 & 2ab + b^2 \\
\hline
2a + 2b - c & -2ac - 2bc + c^2 \\
 & -2ac - 2bc + c^2 \\
\hline
 & * \quad * \quad *
\end{array}
$$

4 What is the square root of $4x^2 + 12xy + 9y^2$?

5. Extract the square root of

$$a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$$

6. What is the square root of

$$x^4 - 4x^3 + 6x^2 - 4x + 1?$$

7. Required the square root of

$$4x^6 + 12x^5 + 5x^4 - 2x^3 + 7x^2 - 2x + 1$$

8. What is the square root of

$$1 - 4a + 4a^2 + 2x - 4ax + x^2?$$

9. Extract the square root of

$$x^6 - 2x^5 + 3x^4 - 2x^3 + x^2.$$

10. What is the square root of

$$x^4 + 4x^2y + 4y^2 - 4x^2 - 8y + 4?$$

11. Extract the square root of

$$9a^4b^2 - 30a^2b + 12a^4b^4 + 25 - 20a^2b^3 + 4a^4b^6.$$

12. What is the square root of

$$4 + 4a^2b + a^4b^2 - 12y^2 - 6a^2by^2 + 9y^4?$$

13. Required the square root of

$$4a^2 - 4ab + b^2 + 12ac - 6bc + 9c^2.$$

14. What is the square root of

$$x^6 + 4x^5 + 10x^4 + 20x^3 + 25x^2 + 24x + 16?$$

15. Find the second root of

$$4a^6 - 4a^4 + 12a^3 + a^2 - 6a + 9.$$

16. Extract the square root of $\frac{a^2 + 2ab + b^2}{c^2 - 2cd + d^2}$.

Extract the square root of the numerator and of the denominator, separately, as in simple quantities.

17. Required the second root of $\frac{16x^2 + 48xy + 36y^2}{9a^2 - 12ac + 4c^2}$

18. Required the square root of $\frac{25a^2 - 90a + 81}{36 + 24x + 4x^2}$.

SECTION IV

To extract the Square Root of a Number.

The following Rule for the extraction of the square root of a number, is derived directly from the method of extracting the square root of an algebraic quantity:

1. *Separate the given number into periods of two figures each, beginning at the right hand. The period on the left may contain either one or two figures.*

2. *Find the greatest second power contained in the left-hand period, and write its root in the quotient's place.*

3. *Subtract the square of this root from the first period of the given quantity, and bring down the next two figures for a dividend.*

4. *Double the root already found, and put it in the divisor's place.*

5. *See how many times the divisor is contained in the new dividend, rejecting the right-hand figure; and annex the quotient both to the root already found ana to the divisor.*

6. *Multiply the divisor, thus increased, by the figure of the root last found, and subtract the product from the dividend.*

7. *Bring down the next two figures of the given number for a new dividend.*

8. *Double the whole root for a new divisor, and divide as before.*

9. *Proceed in this way, until the entire root of the given quantity is extracted.*

Should the divisor not be contained in the dividend, 0 must be annexed both to the root and the divisor; the next two figures must then be brought down, and a new trial made.

The only important point, in which this rule differs from the one given for extracting the square root of an algebraical quantity, is that which requires the given number to be separated into periods of two figures each. This is done to ascertain how many figures the required root will consist of. As the square root of 100 is 10, the square root of every number less than 100 must be less than 10, and consist, of course, of but one figure. And as the square root of 10000 is 100, the square root of every number less than 10000, and more than 100, must be composed of two figures. So, also, as the square root of 1000000 is 1000, the square root of every number smaller than 1000000 must be less than 1000, and consist of not more than three figures. Hence it is, that the given number is separated into periods of two figures each.

1 What is the square root of 106929?

```
         .  .  .
        106929 ( 327.
         9
      --------
  62  | 169
      | 124
      ---------
 647  | 4529
      | 4529
      ---------
         *
```

By separating the number, given in the question, into periods of two figures each, we find that the root will consist of three figures. The greatest even power in the first period, 10, is 9, the square root of which is 3. This root is, therefore, put in the quotient's place; its second power is subtracted from the first period, 10; and the next period, 69, is brought down and annexed to the remainder, for a dividend. This dividend (the last figure, 9, being omitted) is divided by twice the root already found, 6; and the quotient, 2, which is the second figure of the root, is annexed to 6, to complete the divisor. The divisor, thus increased, is multiplied by 2, and the product is subtracted from the dividend. The next period, 29, is annexed to the remainder, and the work is continued as before.

2. What is the square root of 43264?

$$
\begin{array}{r|l}
 & \dot{4}3\dot{2}6\dot{4}\ (\ 208. \\
 & \underline{4} \\
408 & 3264 \\
 & \underline{3264} \\
 & \ \ *
\end{array}
$$

In this example, the divisor, 4, is not contained in the dividend after the right-hand figure is rejected; a zero is, therefore, added to both the root and the divisor, and the next period, 64, is brought down to complete the dividend.

3. What is the square root of 148225?

4. What is the square root of 1522756?

5. Required the square root of 5499025.

6. What is the square root of 28153636?

7. Extract the square root of 9247681.

8. What is the square root of $\frac{9}{16}$? Ans. $\frac{3}{4}$

We find the root of a fraction by extracting the root of the numerator and of the denominator separately.

9. Required the square root of $\frac{25}{49}$?

10. What is the square root of $3\frac{13}{36}$? Ans. $1\frac{5}{6}$.

The mixed number, $3\frac{13}{36}$, reduced to an improper fraction, becomes $\frac{121}{36}$, the square root of which is $\frac{11}{6}$, or $1\frac{5}{6}$, as given above.

11. What is the square root of $5\frac{19}{25}$?

12. Extract the square root of $7\frac{1}{9}$.

13. Required the square root of $30\frac{1}{4}$.

14. What is the square root of 27699169?

15. Required the square root of $\frac{64}{81}$.

16. Find the square root of $13\frac{124}{625}$.

17. What is the square root of $\frac{529}{1225}$?

18. Extract the square root of 3637175481

SECTION V.

Approximate Roots.

1. What is the square root of 7?

Since the second power of 2 is 4, and the second power of 3 is 9, the square root of 7 must be more than 2, and less than 3. Therefore, 7 is not a complete power, and its *exact* root cannot be found; but, by annexing two zeroes to each of the successive re

mainders, we can obtain its *approximate* root, in decimals, to any assignable degree of accuracy. Two zeroes are annexed for each new decimal of the root, for the same reason that two figures are brought down, when the operation is confined to whole numbers. Of course, the decimal places of the root will be equal to half the number of zeroes used.

$$\begin{array}{rl}
 & 7\ (\ 2.645\ \&c \\
 & 4 \\ \hline
46 & \left|\ \begin{array}{l} 300 \\ 276 \end{array}\right. \\ \hline
524 & \left|\ \begin{array}{l} 2400 \\ 2096 \end{array}\right. \\ \hline
5285 & \left|\ \begin{array}{l} 30400 \\ 26425 \end{array}\right. \\ \hline
 & \quad 3975
\end{array}$$

If there were no remainder, the square root of 7 would be $2\frac{645}{1000}$; but, as there is a remainder, it is more than $2\frac{645}{1000}$, although less than $2\frac{646}{1000}$. Therefore, $2\frac{645}{1000}$ differs less than the thousandth part of one from the true root required. By annexing additional zeroes, and continuing the work, we may obtain the root still more accurately.

2. Required the square root of 5.
3. What is the square root of 2?
4. Extract the square root of 823.
5. Find the square root of 527.

6. What is the square root of $\frac{4}{9}$?

The square root of 9 is 3; but, since the numera-

tor is not a complete power, the root whose second power is nearest to 5 must be taken, which is 2. The difference between $\frac{2}{3}$ and the true square root of $\frac{5}{9}$, is less than $\frac{1}{3}$ of a unit.

When both the numerator and the denominator of a fraction are multiplied by the same quantity, its value is not altered; and if both terms of this fraction be multiplied by any perfect power, a nearer approximation to the true root will be obtained. Let them be multiplied by 9.

$$\frac{5\times 9}{9\times 9}=\frac{45}{81}.$$

The square root of 81 is 9, and the nearest square root of 45 is 7; so that $\frac{7}{9}$ expresses the value of $\sqrt{\frac{5}{9}}$ within $\frac{1}{9}$ part of a unit.

Again, let $\frac{45}{81}$ be multiplied by 144, which is the second power of 12.

$$\frac{45\times 144}{81\times 144}=\frac{6480}{11664}.$$

The square root of $\frac{6480}{11664}$ is $\frac{80}{108}$ nearly; consequently, $\frac{80}{108}$ differs less than $\frac{1}{108}$ of a unit from the value of the exact square root of $\frac{5}{9}$.

In general, the larger the power by which the terms of a fraction are multiplied, the nearer to the true root will be the approximation.

7. What is the square root of $3\frac{1}{7}$?

Reduce this mixed number to an improper fraction, and proceed as before.

$$3\frac{1}{7}=\frac{22}{7}, \text{ and } \frac{22\times 7}{7\times 7}=\frac{154}{49}.$$

The nearest square root of $\frac{144}{49}$ is $\frac{12}{7}$ or $1\frac{5}{7}$, which is the root required within less than $\frac{1}{7}$ of one

8. Required the square root of $\frac{3}{5}$.
9. What is the square root of $5\frac{4}{11}$?
10. Find the square root of $\frac{7}{8}$.
11. Required the square root of $2\frac{2}{3}$.

12. What is the square root of $a^2 + x^2$?

It has been shown, already, that no binomial is a perfect second power. The approximate root of a surd can be found by the common rule for extracting the square root of a compound quantity, thus:

$$a^2 + x^2 \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}\ \&c \right.$$

$$\begin{array}{l|l}
 & a^2 \\
\hline
2a + \frac{x^2}{2a} & x^2 \\
 & x^2 + \frac{x^4}{4a^2} \\
\hline
2a + \frac{x^2}{a} - \frac{x^4}{8a^3} & -\frac{x^4}{4a^2} \\
 & -\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6} \\
\hline
2a + \frac{x^2}{a} - \frac{x^4}{4a^3} + \frac{x^6}{16a^5} & \frac{x^6}{8a^4} - \frac{x^8}{64a^6} \\
 & \frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{x^{10}}{64a^8} + \frac{x^{12}}{256a^{10}} \\
\hline
2a + \frac{x^2}{a} - \frac{x^4}{4a^3} + \frac{x^6}{8a^5} - \frac{5x^8}{128a^7} & -\frac{5x^8}{64a^6} + \frac{x^{10}}{64a^8} - \frac{x^{12}}{256a^{10}}
\end{array}$$

13. Required the square root of $1 + x$.
14. What is the square root of $x^2 - z^2$?
15. Extract the square root of $a^4 + 1$.
16. What is the square root of $\frac{5}{12}$?
17. Extract the square root of 7641.

CHAPTER XI

EQUATIONS OF THE SECOND DEGREE

SECTION I.

Pure Equations.

An equation of the second degree contains the second power of the unknown quantity. When the unknown quantity appears *only* in the second power, the equation is said to be pure.

1. What number is that, which, being multiplied by itself, and the product doubled, will give 162?

Let $x =$ the number.
Then $2x^2 = 162$, by the question;
and $x^2 = 81$, by division.
$x = 9$.

Ans. 9.

2. A farmer, being asked how many cows he had, answered, that if the number were multiplied by 5 times itself, the product would be 720. How many had he?

Let $x =$ the number of cows.
Then $5x^2 = 720$, by the question.
$x^2 = 144$, and $x = 12$.

Ans. 12 cows.

3. A gentleman, being asked the price of his hat, answered, that if it were multiplied by itself, and 26 were subtracted from the product, the remainder multiplied by 5 would be 190. What was the price of the hat?

$$\text{Let } x = \text{the price of the hat.}$$
$$\text{Then } 5\,x^2 - 130 = 190, \text{ by the question.}$$
$$5\,x^2 = 190 + 130, \text{ or } 320, \text{ by transposition}$$
$$x^2 = 64, \text{ and } x = 8.$$

ANS. $8.

4. A gentleman, being asked the age of his son replied, that if from the square of his age were subtracted his own age, which was 30 years, and the remainder were multiplied by his son's age, the product would be 6 times his age. How old was he?

$$\text{Let } x = \text{the son's age.}$$
Then, by the conditions of the question,
$$(x^2 - 30)\,x, \text{ or } x^3 - 30\,x = 6\,x.$$
$$x^2 - 30 = 6, \text{ by dividing by } x.$$
$$x^2 = 36, \text{ and } x = 6.$$

ANS. 6 years.

5. What two numbers are those, which are to each other as 3 to 4, and the difference of whose squares is 112?

$$\text{Let } x = \text{the larger number,}$$
$$\text{and } \frac{3\,x}{4} = \text{the smaller.}$$
$$\text{Then } x^2 - \frac{9\,x^2}{16} = 112,$$
$$\text{or } 16\,x^2 - 9\,x^2 = 1792, \text{ by multiplication.}$$

ANS. 16 and 12.

6. There is a certain room, the sum of whose length and width is to its length as 5 to 3; and the same sum, multiplied by the length, is equal to the width multiplied by 60. What are the dimensions of the room?

Let $x =$ the length,
and $y =$ the width of the room.

A. Then $x + y = \frac{5x}{3}$, by the question,

B. and $60\,y = x^2 + x\,y$, that is, $(x + y)\,x$

C. $y = \frac{2x}{3}$, by reducing equation A.

D. $y = \frac{x^2}{60 - x}$, by reducing equation B.

E. $\frac{x^2}{60 - x} = \frac{2x}{3}$, by comparing equations C and D.

$$3\,x^2 = 120\,x - 2\,x^2,$$

and $3\,x = 120 - 2\,x$, by dividing by x.

$$3\,x + 2\,x = 120, \text{ and } x = 24.$$

$$y, \text{ or } \frac{2x}{3} = 16.$$

ANS. Length, 24 feet;
Width, 16 feet.

From these operations may be derived the following RULE for solving pure equations of the second degree:

Find the value of the second power of the unknown quantity, in the same manner as the value of the unknown quantity is found in simple equations; and then extract the square root of each member of the equation.

Sometimes, as in the last question, the second power can be made to disappear by division.

7. A boy bought a number of oranges for 36 cents;

and the price of an orange was to the number bought as 1 to 4. How many oranges did he buy, and what did he give apiece?

8. A merchant sold a quantity of flour for a certain sum, and at such a rate, that the price of a barrel was to the number of barrels as 4 to 5: if he had received 45 dollars more for the same quantity, the price of a barrel would have been to the number of barrels as 5 to 4. How many barrels did he sell, and at what price?

9. A gentleman exchanges a field, 81 rods long and 64 rods wide, for an equal quantity of land in the form of a square. What was the side of the square?

10. How long and wide is a rectangular field containing 864 rods, the width of which is equal to $\frac{2}{3}$ of the length?

11. A certain street contains 144 rods of land; and if the length of the street be divided by its width, the quotient will be 16. How long and wide is the street?

12. A trader sold two pieces of broadcloth, which together measured 18 yards; and he received as many dollars a yard for each piece as it contained yards. Now, the sums received for the two were to each other as 25 to 16. How many yards were there in each piece?

Let $x =$ the yards in the longer piece,
and $y =$ the yards in the other.

$$\text{Then } x + y = 18;$$
$$\text{and } x = 18 - y.$$

$$\text{Again, } x^2 = \text{the whole price of one,}$$
$$\text{and } y^2 = \text{the price of the other.}$$
$$\text{Therefore, } x^2 = \frac{25\,y^2}{16}, \text{ by the question;}$$
$$\text{and } x = \frac{5\,y}{4}, \text{ by evolution.}$$
$$\frac{5\,y}{4} = 18 - y, \text{ by comparing the values of } x.$$

ANS. 10 yards; 8 yards.

13. A man divided 14 dollars between his son and daughter in such a manner, that the quotient of the daughter's part divided by the son's, was $\frac{9}{16}$ of the son's part divided by the daughter's. What was the share of each?

14. A house contains two square rooms, the areas of which are to each other in the proportion of 25 to 9; and a side of the larger room exceeds a side of the smaller by 10 feet. What are the dimensions of the rooms?

15. In a certain orchard there are 4 more rows of trees than there are trees in a row; and if the same number of trees were so arranged that there should be 64 added to each row, the number of the rows would be reduced to 4. How many trees are there in the orchard?

$$\text{Let } x = \text{the trees in a row.}$$
$$\text{Then } x + 4 = \text{the number of rows,}$$
$$\text{and } x^2 + 4\,x = \text{the number of trees.}$$
$$\left.\begin{array}{r}\text{Also } (x + 64) \times 4,\\ \text{or } 4\,x + 256,\end{array}\right\} = \text{the number of trees.}$$
$$\text{Then } x^2 + 4\,x = 4\,x + 256.$$

ANS. 320 trees

16. When an army was formed in solid column, there were 9 more men in file than in rank; but when it was formed in 9 lines, each rank was increased by 900 men. Of how many men did the army consist?

17. A gentleman has two squares of shrubbery in his grounds, the difference of whose sides is to the side of the greater square as 2 to 9; and the difference of their areas is 128 yards. What are the sides of the squares?

18. Says A to B, "Our ages are the same; but if I were 5 years older, and you were 5 years younger, the product of our ages would be 96." What are their ages?

19. What number is that, which being added to 10 and subtracted from 10, the product of the sum, multiplied by the difference, will be 51?

20. There is a rectangular field, whose length is to its breadth in the proportion of 6 to 5. A part of this, equal to $\frac{1}{6}$ of the whole, being an orchard, there remain for tillage 625 square rods. What are the length and breadth of the field?

21. It requires 108 square feet of carpeting to cover a certain entry; and the sum of its length and breadth is equal to twice their difference. How long and wide is it?

22. Required two numbers which are to each other as 1 to 3, and the sum of whose second powers is equal to 5 times the sum of the numbers.

23. The area of an oblong room is 400 square feet; and if its width were equal to its length, its area would

be $\frac{2}{16}$ greater. What are the dimensions of the room?

24. A charitable person distributed a certain sum among some poor men and women, the numbers of whom were in the proportion of 4 to 5. Each man received $\frac{1}{8}$ as many shillings as there were persons relieved; and each woman received twice as many shillings as there were women more than men. The men received, altogether, 18 shillings more than the women. How many were there of each?

25. A gentleman, being asked the ages of his two sons, replied, that they were to each other as 3 to 4; and that the product of their ages was 48. What were their ages?

26. A gentleman has an oblong garden of such dimensions, that if the difference of the sides be multiplied by the greater side, the product will be 40 square rods; but if the difference be multiplied by the shorter side, the product will be 15 rods. What are the length and width of the garden?

Let $x =$ the less side,
and $y =$ the difference of the sides.
Then $x + y =$ the greater side.

$$\left.\begin{aligned} \text{Therefore, } x\,y + y^2 &= 40, \\ \text{and } x\,y &= 15, \end{aligned}\right\} \text{by the conditions of the question.}$$

$$x = \frac{15}{y}.$$

By substituting the value of x in the first equation, we shall obtain

$$15 + y^2 = 40,$$
$$\text{and } y = 5.$$

SECTION II.

Affected Equations.

As a *pure* equation of the second degree contains the unknown quantity only in the form of its second power, all the terms in which it appears can be united in one term, whose root, as we have seen, can be readily found.

An affected equation of the second degree contains not only the square of the unknown quantity in one term, but also the unknown quantity itself in another term.

Thus, $x^2 + 4x = 77$ is an affected equation of the second degree, in which the unknown quantity appears in two terms; for x^2 and x cannot be actually added together so as to make but one term.

When an equation of this sort is formed, it may contain the unknown quantity in any number of terms, provided it be only in the first and second powers; for, in this case, the terms may all be reduced to two.

Thus, if we have the equation

$$5x^2 + 8x - 24 + x^2 - 3x = 4x^2 + x + 48,$$

by transposition we obtain

$$5x^2 + x^2 - 4x^2 + 8x - 3x - x = 48 + 24;$$

and, by adding the similar terms,

$$2x^2 + 4x = 72;$$

and, by dividing all the terms by 2,

$$x^2 + 2x = 36.$$

Affected and pure equations of the second degree

are solved in the same manner: in both, *we find the value of the unknown quantity by extracting the square root of each member of the equation.* When the member containing the unknown quantity is a complete power, the process is as direct and simple in an affected as in a pure equation.

1. Given $x^2 + 2\,a\,x + a^2 = b^2$. What is the value of x? Ans. $b - a$.

In this question, a and b represent known quantities. Now, we know by inspection, that the first member of the equation, $x^2 + 2\,a\,x + a^2$, is the complete second power of the binomial quantity $x + a$; and the other member, b^2, is the second power of b. By extracting the square root of each member, therefore, we obtain the equation,

$$x + a = b.$$

And $x = b - a$, by transposition.

If we suppose $a = 4$ and $b = 9$, by raising $x + 4$ and 9 to their second powers, instead of the above equation, we shall have

$$x^2 + 8\,x + 16 = 81,$$

$$\text{or } \sqrt{x^2 + 8\,x + 16} = \sqrt{81}.$$

$$\text{That is, } x + 4 = 9,$$

and $x = 9 - 4 = 5$, by transposition.

2. Given $x^2 + 2\,a\,x = b$. What is the value of x? Ans. $\sqrt{b + a^2} - a$.

In this question, the member of the equation containing the unknown quantity, is not a complete power;

and, of course, while it remains in its present form, its root, and consequently the value of the unknown quantity, cannot be found. But it is possible to add such a quantity to the first member as shall make it a perfect square. The necessary quantity must be added to *both* members of the equation, to preserve its equality.

But how shall we find the quantity which must be added to complete the square? If we examine the equation given above,

$$x^2 + 2\,a\,x = b,$$

we shall be satisfied that the root of the first member, whatever it may be, is not a single term; for any power of one term consists of but one term. But if its root consist of two terms, one term is wanting to complete the square; for the second power of a binomial quantity contains three terms, [See Chap. VII. Sec. III.] whereas the given quantity contains but two.

The terms given, $x^2 + 2\,a\,x$, are the first two terms of the second power of the binomial $x + a$; and the third term of the same quantity is a^2. [See Chap. IX. Ques. 18.] By adding this quantity, therefore, to both of the given members, we make the first member a complete square; and the equation becomes

$$x^2 + 2\,a\,x + a^2 = b + a^2.$$

And $\sqrt{x^2 + 2\,a\,x + a^2} = \sqrt{b + a^2}$,

or $x + a = \sqrt{b + a^2}$, by evolution,

and $x = \sqrt{b + a^2} - a$, by transposition.

Observe that the coefficient of the second term

given is $2a$, one half of which is $1a$ or a; and that the quantity added to complete the square, a^2, is the second power of a.

If, in the last example, we suppose $a = 4$, and $b = 84$, the given equation will be

$$x^2 + 8x = 84.$$

Now, if we add, as before, the second power of the value of a, $(4 \times 4 = 16,)$ to both sides of the equation, the first member will be a complete square; and the equation will be solved thus ·

$$x^2 + 8x + 16 = 84 + 16, \text{ or } 100.$$

$$\text{And } \sqrt{x^2 + 8x + 16} = \sqrt{100},$$

$$\text{or } x + 4 = 10;$$

$$\text{and } x = 10 - 4, \text{ or } 6.$$

Observe that 16, which is added to complete the square, is the second power of 4, or half of the coefficient of the second term of the given equation.

Hence, to render the first member a complete second power, *we add the square of half the coefficient of the second term to both members of the equation.*

3. Given the equation $87 + 7x^2 - 123 + 3x = 5x^2 + 118 - 5x$. What is the value of x?

$7x^2 - 5x^2 + 3x + 5x = 118 + 123 - 87$, by transposition.

$2x^2 + 8x = 154$, by uniting terms,

and $x^2 + 4x = 77$, by division.

$x^2 + 4x + 4 = 77 + 4$, or 81, by completing the square

$x + 2 = 9$, by evolution,

and $x = 9 - 2$, or 7. Ans. 7

4. Given the equation $3x^2 + 89 + \frac{x^2}{3} + 20x = 224 + 3x^2 + 8x$; to find the value of x.

$3x^2 + \frac{x^2}{3} - 3x^2 + 20x - 8x = 224 - 89$, by transposition.

$\frac{x^2}{3} + 12x = 135$, by uniting terms.

$x^2 + 36x = 405$, by removing the denominator.

$x^2 + 36x + 324 = 405 + 324$, or 729, by completing the square.

$x + 18 = 27$, by evolution.

$x = 27 - 18$, or 9.

5. What is the value of x in the equation $3x^2 + 2x = 161$?

$$3x^2 + 2x = 161.$$

$x^2 + \frac{2x}{3} = \frac{161}{3}$, by removing the coefficient of x^2.

$x^2 + \frac{2x}{3} + \frac{1}{9} = \frac{161}{3} + \frac{1}{9} = \frac{484}{9}$, by completing the square.

In this equation, the second term has a fractional coefficient; for $\frac{2x}{3}$ is the same as $\frac{2}{3}x$. The half of $\frac{2}{3}$ is $\frac{1}{3}$, the second power of which is $\frac{1}{9}$. The fractions of the second member, $\frac{161}{3}$ and $\frac{1}{9}$, are reduced to a common denominator, and added together in the usual way.

6. Given $5x^2 + 3x = 344$. What is the value of x?

$$5x^2 + 3x = 344.$$

$x^2 + \frac{3x}{5} = \frac{344}{5}$, by removing the coefficient of x^2.

$x^2 + \frac{3x}{5} + \frac{9}{100} = \frac{344}{5} + \frac{9}{100}$, or $\frac{6889}{100}$

Half of $\frac{3}{5}x$ is $\frac{3}{10}x$; and the second power of $\frac{3}{10}$ is $\frac{9}{100}$.

7. Required the value of x in the following equation; $x^2 - 14x = 51$.

$$x^2 - 14x = 51.$$

$$x^2 - 14x + 49 = 100, \text{ by completing the square.}$$

$$x - 7 = 10, \text{ by evolution;}$$

$$\text{and } x = 10 + 7, \text{ or } 17.$$

In this operation, as the second term of the first member, $-14x$, is a negative quantity, the second term of the root must also be negative. It should not be forgotten, that the square root of a positive quantity may be either positive or negative.

It will be observed, that *the square root of the first member of the equation always consists of the unknown quantity, and half the coefficient of the second term.* In practice, therefore, it is not necessary to complete the square of that member, and then extract the root.

8. Divide 34 into two such numbers that their product shall be 225.

$$\text{Let } x = \text{one number.}$$

$$\text{Then } 34 - x = \text{the other,}$$

$$\text{and } x(34 - x) = \text{their product.}$$

$$\text{Therefore, } 34x - x^2 = 225, \text{ by the question.}$$

It will be remembered, that there can be no such quantity as the root of $-x^2$. [See Chap. X. Sec II.] Consequently, in the second power of a binomial or residual quantity, the first and last terms must always be positive, as each is the second power of one term of the root. Therefore, the first member of the last equation,

$$-x^2 + 34x = 225,$$

is not, in its present form, a part of the second power of any binomial whatever. *The signs of all the terms may be changed, however, without affecting the equality of the members;* and, when this has been done, the square of the first member can be completed in the usual manner.

$$x^2 - 34x = -225, \text{ by changing all the signs.}$$
$$x^2 - 34x + 289 = -225 + 289, \text{ or } 64.$$
$$x - 17 = \mp 8, \text{ by evolution,}$$
$$\text{and } x = \mp 8 + 17.$$

ANS. 25 and 9.

From these examples and observations, we derive the following RULE for solving affected equations of the second degree:

1. *Collect all the terms containing the unknown quantity in one member of the equation, and all the known quantities in the other.*

2. *Arrange the terms that contain the unknown quantity according to their powers, as in Division.*

3. *If the square of the unknown quantity have a coefficient or a divisor, it should be removed in the usual manner.*

4. *If the term containing the square of the unknown quantity be negative, make it positive by changing the signs of all the terms.*

5. *Complete the square of the first member, by adding the square of half the coefficient of the second term to both sides of the equation.*

6. *Reduce the equation by extracting the square root*

of each member, and transposing the known part of the binomial.

9. Required two numbers, whose difference shall be 8 and their product 105.

$$\text{Let } x = \text{the smaller number,}$$
$$\text{and } x + 8 = \text{the larger.}$$
$$\text{Then } x^2 + 8x = 105, \text{ by the question.}$$
$$x + 4 = \sqrt{105 + 16}, \text{ or } 11.$$

10. What two numbers are there whose sum is 30, and whose product is equal to eight times their difference?

$$\text{Let } x = \text{the greater number.}$$
$$\text{Then } 30 - x = \text{the less,}$$
$$\text{and } 2x - 30 = \text{their difference.}$$
$$\text{Therefore, } x(30 - x) = 8(2x - 30), \text{ by the question,}$$
$$\text{or } 30x - x^2 = 16x - 240;$$
$$\text{and } x^2 - 30x = -16x + 240, \text{ by changing the signs.}$$
$$x^2 - 14x = 240, \text{ by transposition,}$$
$$\text{and } x^2 - 14x + 49 = 240 + 49, \text{ or } 289.$$

Another Way to complete the Square.

11. Given $x^2 = c - bx$; to find the value of x.

$$\text{A. } x^2 + bx = c, \text{ by transposition.}$$
$$x^2 + bx + \frac{b^2}{4} = c + \frac{b^2}{4}, \text{ by completing the square.}$$

Let this equation be freed from fractions, and it will become

$$\text{C. } 4x^2 + 4bx + b^2 = 4c + b^2$$

The first member of this equation being a perfect square, its root may be found in the usual manner.

$$\sqrt{4x^2 + 4bx + b^2} = \sqrt{4c + b^2},$$
or $2x + b = \sqrt{4c + b^2}$, by evolution,
and $2x = \sqrt{4c + b^2} - b$, by transposition.

ANS. $\sqrt{\frac{4c+b^2}{4}} - \frac{b}{2}$.

We might have obtained equation C directly from equation A, by multiplying it by 4 and adding the square of b, the coefficient of the second term, to each member.

12. Required the value of x in the following equation; $ax^2 - c = -bx$.

A. $ax^2 + bx = c$, by transposition.

B. $x^2 + \frac{bx}{a} = \frac{c}{a}$, by removing the coefficient of x^2.

C. $x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{c}{a} + \frac{b^2}{4a^2}$, by completing the square

If we remove the denominators from this equation, we shall obtain

D. $4a^2x^2 + 4abx + b^2 = 4ac + b^2$,
and $\sqrt{4a^2x^2 + 4abx + b^2} = \sqrt{4ac + b^2}$,
or $2ax + b = \sqrt{4ac + b^2}$, by evolution,
and $2ax = \sqrt{4ac + b^2} - b$, by transposition.

ANS. $\sqrt{\frac{4ac+b^2}{4a^2}} - \frac{b}{2a}$.

Compare equations A and D. It will be seen that if we multiply the former by $4a$, that is, by 4 times the coefficient of x^2, and add to each member the

second power of b, which is the coefficient of x, we shall obtain the latter equation.

From these operations may be derived the following RULE for completing the square, which will be found more simple and convenient than that already given, where the coefficient of x^2 is a *small* number, and where the coefficient of x is an *odd* number; as it does not introduce fractions into the operation.

Having prepared and arranged the equation as before, multiply both members by four times the coefficient of the first term; that is, of the term which contains the square of the unknown quantity. If it have no coefficient, multiply by four.

Add the square of the coefficient of the second term to both sides of the equation; and extract the square root of each member.

13. Given $x^2 + 5x = 126$; to find the value of x.

As the first term, x^2, has no coefficient, the equation must be multiplied by 4.

$$4x^2 + 20x = 504.$$

Next, add the square of 5, the coefficient of the second term, to both members.

$$4x^2 + 20x + 25 = 504 + 25, \text{ or } 529.$$

The first member being now a complete square, extract the root as before.

$$\sqrt{4x^2 + 20x + 25} = \sqrt{529},$$
$$\text{or } 2x + 5 = 23.$$
$$2x = 23 - 5, \text{ or } 18, \text{ by transposition,}$$
$$\text{and } x = 9.$$

14. Required the value of x in the following equation; $3x^2 + 3x = 216$.

Multiply by 4 times the coefficient of x^2, that is, by $4 \times 3 = 12$.

$$36x^2 + 36x = 2592.$$

Add the square of 3, the coefficient of x.

$$36x^2 + 36x + 9 = 2592 + 9, \text{ or } 2601.$$

$$\sqrt{36x^2 + 36x + 9} = \sqrt{2601},$$

$$\text{or } 6x + 3 = 51.$$

$$6x = 51 - 3, \text{ or } 48; \text{ and } x = 8.$$

15. What is the value of x in the following equation; $3x^2 - 19x = -6$?

$$3x^2 - 19x = -6.$$

$$36x^2 - 228x = -72, \text{ by multiplying by } 12.$$

$$36x^2 - 228x + 361 = 289, \text{ by adding } 19^2.$$

16. Given $\frac{4x^2}{3} - 11 = \frac{x}{3}$; to find the value of x.

$$\frac{4x^2}{3} - 11 = \frac{x}{3}$$

$$4x^2 - 33 = x, \text{ by removing the denominators.}$$

$$4x^2 - x = 33, \text{ by transposition.}$$

$$64x^2 - 16x = 528, \text{ by multiplying by } 16.$$

$$64x^2 - 16x + 1 = 528 + 1, \text{ or } 529, \text{ by completing the square.}$$

17. There are two numbers whose difference is 9, and $\frac{1}{2}$ their product is 10 more than the square of the smaller number. What are the numbers?

18. The length of a room exceeds its width by 9 feet; and its area is 400 feet. What are the dimensions of the room?

SECTION III

Questions producing Affected Equations.

1. The ages of a man and his wife amount to 42 years, and the product of their ages is 432. What is the age of each?

2. A gentleman, being asked the ages of his son and daughter, replied, that his son was 5 years older than his daughter, and that the product of their ages was 266. What were their ages?

3. The length of a room exceeds its width by 8 feet, and its area is 768 feet. What are its length and width?

4. The difference of two numbers is 6; and the square of the greater exceeds twice the square of the less by 47. Required the numbers.

5. A gentleman divided 28 dollars between his two sons in such a manner, that the product of their shares was 192. What was the share of each?

6. The wall which encloses a rectangular garden, is 128 yards long, and the area of the garden is 1008 yards. What are its length and breadth?

7. A man bought a certain number of sheep for 80 dollars. If he had bought 4 more for the same money, they would have come to him a dollar apiece cheaper. How many did he buy?

8. It is required to find two numbers, whose sum shall be 14; and such, that 18 times the greater shall be equal to 4 times the square of the less.

9. A man paid 120 dollars more for his watch than for a chain; and the price of the watch was to that of the chain as the price of the chain was to 10. Required the price of each.

10. In a parcel containing 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins; and each copper coin is worth as many pence as there are silver coins; and the whole is worth 18*s*. How many coins are there of each sort?

11. A drover bought a number of oxen for 675 dollars; which he sold again for 48 dollars a head; and he gained, by the bargain, as much as he gave for one ox. How many oxen did he buy?

12. Two travellers, A and B, set off at the same time to a place distant 150 miles. A travels 3 miles an hour faster than B, and arrives at his journey's end 8 hours and 20 minutes before him. How many miles did each travel per hour?

13. What two numbers are there, whose sum is 25 and product 144?

14. The age of A is 12 years more than that of B; and the product of their ages is 640. What is the age of each?

15. The sum of two numbers is 30; and if 18 be added to $\frac{1}{3}$ of their product, the sum will be equal to the square of the smaller number. What are the numbers?

16. Says A to B, "The product of our ages is 120; and if I were 3 years younger, and you were 2

years older, the product of our ages would still be 120." What are their ages?

17. A farmer sold a certain number of sheep for £120. If he had sold 8 more for the same money, he would have received 10 shillings less for each sheep. How many did he sell?

18. Two benevolent gentlemen, A and B, distributed each 1200 dollars among a certain number of poor persons. A relieved 40 persons more than B; but B gave 5 dollars more to each person than A. How many persons did each relieve?

19. A person bought two pieces of cloth; the finer of which, at 4 shillings a yard more than the other, cost £18. But the coarser piece, which was 2 yards longer than the finer, cost only £16. How many yards were there in each piece? and what was the price of a yard of each?

20. An officer would arrange 1200 men in a solid body, so that each rank may exceed each file by 59 men. How many must be placed in rank and file?

21. In an orchard containing 900 trees, the trees are so planted that there are 11 more rows than there are trees in a row. Required the number of rows; also the number of trees in a row.

22. The perimeter of a room is 48 feet; and the area of the floor is equal to 35 times the difference of its length and breadth. What are the dimensions of the room?

23. A drover bought a number of sheep for 190 dollars. Having lost 8 of them, he must sell the remainder at a profit of 8 shillings apiece, not to lose

money by the bargain. How many sheep did he buy? and at what price?

24. A merchant sold a quantity of sugar for £56, by which he gained as much per cent. as the whole cost him. How much did it cost?

Let $x =$ the cost of the sugar.

Then $56 - x =$ the gain.

Again, $\frac{x}{100} =$ the rate per cent.;

and $\frac{x^2}{100} =$ the gain.

Therefore, $\frac{x^2}{100} = 56 - x$.

25. A trader sold a quantity of flour for 39 dollars, and gained as much per cent. as the flour cost him. What did he give for the flour?

26. A butcher bought a certain number of calves for 200 dollars; and, reserving 15, he sold the rest for 180 dollars, by which he gained 2 shillings a head. How many calves did he buy? and at what price?

27. A grass-plot, 18 yards long and 12 wide, is surrounded by a border of flowers of a uniform width. The areas of the grass-plot and border are equal What is the width of the border?

28. A square court-yard has a gravel walk round it. The side of the court wants 2 yards of being 6 times the breadth of the walk; and the number of square yards in the walk, exceeds the number of yards in the periphery of the court by 164. What is the area of the court?

29. If the square of a certain number be taken from 40, and the square root of their difference be

plied by the number that I have, the product will be 266." How many dollars has each?

39. A man has three children, A, B and C; A being the oldest, and C the youngest. Now, the difference of A and B's ages exceeds the difference of B and C's by 6 years. The sum of all their ages is 33 years, and the sum of their squares 467. Required their ages.

$$\text{Let } x = \text{B's age,}$$
$$\text{and } y = \text{the difference of B and C's ages.}$$
$$\text{Then } x - y = \text{C's age,}$$
$$\text{and } x + y + 6 = \text{A's age.}$$
$$x + x - y + x + y + 6 = 33, \text{ by the question,}$$
$$\text{or } 3\,x = 33 - 6, \text{ or } 27,$$
$$\text{and } x = 9.$$

Again, according to the question, the sum of the squares of all their ages is 467; that is,

$$x^2 + (x - y)^2 + (x + y + 6)^2 = 467,$$

or, by involution and addition,

$$3\,x^2 + 2\,y^2 + 12\,x + 12\,y + 36 = 467;$$

and, by substituting the value of x, we have

$$243 + 2\,y^2 + 108 + 12\,y + 36 = 467,$$
$$\text{or } 2\,y^2 + 12\,y = 80, \text{ by transposition.}$$

40. A farmer sold 80 bushels of wheat and 100 bushels of rye for £65; and each at such a rate, that he sold 60 bushels more of rye for £20 than of wheat for £10. What was the price of each?

CHAPTER XII.

MISCELLANEOUS QUESTIONS

1. A GENTLEMAN bought three kinds of wine, of each an equal quantity. For the sherry he gave \$8 a dozen; for the port, \$9; and for the madeira, \$11. The whole came to \$168. How many dozen of each kind did he buy?

2. A steam-boat has 81 passengers; there being twice as many women as children, and three times as many men as women. What is the number of men, women and children?

3. Divide \$9289 between A and B in such a manner, that A's share shall be to B's as 2 to 5.

4. A man, dying, left an estate valued at \$14832. In his will he gave $\frac{1}{2}$ of his property to his wife; and directed the remainder to be so divided between his son and daughter, that the daughter's portion might be to the son's as 3 to 5. What was the share of each?

5. What is that number, to which if you add $\frac{1}{2}$ of itself, and from the sum subtract $\frac{3}{4}$ of itself, $\frac{1}{3}$ of the remainder is 3?

6. A farmer bought 12 sheep, 5 cows, 2 yoke of

oxen and 3 horses, for 795 dollars. A cow cost as much as 6 sheep, a yoke of oxen as much as 3 cows, and a horse as much as 3 oxen. What did he give for each?

7. A man divided a certain sum of money equally between his son and daughter; but had he given his son 33 dollars more, and his daughter 47 dollars less, her share would have been but $\frac{1}{3}$ of his. What was the sum divided?

8. Divide 46 dollars into two such parts, that $\frac{1}{7}$ of one and $\frac{1}{3}$ of the other may be 10 dollars.

9. A man divided 198 acres of land between his three children in such a manner, that A's part was to B's in the ratio of 3 to 8; and C had as many acres as both his brothers. What was the share of each?

10. A man bought a certain quantity of wine for 94 dollars; and after 7 gallons had leaked out, he sold $\frac{1}{4}$ of the remainder, at cost, for \$20. How many gallons did he buy?

11. If a certain number be divided by the sum of its digits, the quotient will be 8; but if the digits be inverted, and that number divided by 2 less than their difference, the quotient will be 9. What is the number?

12. Two friends bought a horse together; and when one had paid $\frac{2}{5}$ and the other $\frac{4}{9}$ of the price agreed upon, they still owed 21 dollars. What was the price of the horse?

13. In a certain university there are 384 students, $\frac{3}{4}$ of whom belong to the academical department; and in the departments of law, divinity and medi-

cine, the students are to each other as the numbers 1, 2 and 3. How many students are there in each department?

14. A man agreed to carry 47 earthen jugs to a certain place. For every one he delivered safe, he was to receive 6 cents; and for every one he broke, he was to pay 10 cents. He received $1,54. How many jugs did he break?

15. Divide $1170 among three persons, A, B and C, in proportion to their ages. Now, B is a third part older than A, and A is half as old as C. What is the share of each?

16. Three men, A, B and C, pay a tax of 594 dollars. The property of A is to that of B as 3 to 5; and the property of B is to that of C as 8 to 7. What part of the tax is paid by each?

17. A father gives to his six sons $2010, which they are to divide according to their ages, so that each elder son shall receive $24 more than his next younger brother. What is the share of the youngest son?

18. If I multiply a certain number by 6, add 18 to the product, and divide the sum by 9, the quotient will be 20. What is the number?

19. Divide 119 into three such parts, that the second divided by the first will give 3 for a quotient, and 3 for a remainder; and the third divided by the second will also give 3 for a quotient, and 3 for a remainder.

20. A school-master, being asked how many dollars he received a month for teaching, replied, "If I add 9 to $\frac{1}{4}$ part of the number of dollars I receive, mul-

tiply the sum thus obtained by 7, subtract 15 from the product, multiply the remainder by 6, and then take away the cipher from the right of the number last obtained, I shall have $54." What were his wages?

21. Some travellers find a purse of money, which they agree to share equally. If they take 5 dollars apiece, one man will receive nothing; but if they take 4 dollars, there will be seven dollars left. What is the number of travellers? What is the sum to be divided?

22. There are two numbers, the product of whose sum multiplied by the greater, is 144; and whose difference, multiplied by the less, gives 14. Required the numbers.

23. A courier had been travelling 4 days, at the rate of 6 miles an hour, when another was sent after him, who travelled 8 miles an hour. In how many days will the second courier overtake the first, if they both travel 15 hours a day?

24. A gentleman has a rectangular garden 36 rods in circumference; and the square of the width is to the square of the length as 16 to 25. Required its dimensions.

25. Two travellers, A and B, began a journey of 300 miles at the same time. A travelled a mile an hour faster than B, and arrived at his journey's end 10 hours before him. How many miles an hour did each travel?

26. Two sportsmen, walking over a marsh, started a flock of plover. The first one fired, and brought

down $\frac{8}{9}$ of the whole flock. Afterwards, the second one fired, and killed a number equal to the square root of half the flock; when only 2 birds were left. How many birds were there in the flock?

27. Required the side of a square field, which shall contain the same quantity of land as another field, which is 72 rods long and 18 rods wide.

28. Three planters, A, B and C, together possess 2658 acres of land. If B sell A 215 acres, then will A's plantation exceed B's by 236 acres; but if B buy $167\frac{1}{2}$ acres of C's plantation, they two will have the same quantity of land. How many acres has each?

29. Required two numbers, whose sum, multiplied by their product, shall be equal to 12 times the difference of their squares; the numbers being to each other in the ratio of 2 to 3.

30. It is required to form a regiment, containing 865 men, into two squares, one of which shall contain 7 more men in rank and file than the other. How many men must each of the squares contain?

31. A man, having travelled 108 miles, found that he could have performed the same journey in 6 hours less, if he had travelled 3 more miles an hour. At what rate did he travel?

32. Two persons, A and B, set out at the same time from two towns, distant 396 miles; and, having travelled as many days as A travelled miles daily more than B, they met each other. It then appeared that A had travelled 216 miles. How many miles did each travel per day?

33. Two merchants, A and B, trade in company

and gain $1930,28. Of the capital employed, A furnished $4000 and B $ 7000. What is each man's share of the gain?

34. A farmer, being asked how many acres of land he owned, answered, that the number was expressed by two digits, whose sum, increased by 7, would be equal to three times the left-hand digit; and he added, that, if he owned 18 acres less, the digits expressing the number would be inverted. How many acres were there in his farm?

35. Several gentlemen made an excursion, each taking the same sum of money. Each had as many servants attending him as there were gentlemen; the number of dollars which each had, was double the number of all the servants, and the whole sum of money taken out, was 3456 dollars. How many gentlemen were there?

36. Four farmers, A, B, C and D, hired a pasture, for which they paid 81 dollars. A put in 4 cows for 3 months; B, 8 cows for 2 months; C, 7 cows for 5 months; and D, 3 cows for 6 months. How much of the rent must each man pay?

37. There is a certain number, the left-hand digit of which is equal to 3 times the right-hand digit; and if 12 be subtracted from the number, the remainder will be equal to the square of the left-hand digit. Required the number.

38. A man has two horses and two saddles, one of which is worth $40, and the other $5. When the best saddle is upon the first horse, and the worst saddle upon the second, the former is worth just twice

as much as the latter; but when the worst saddle is upon the first horse, and the best saddle upon the second, the latter is worth \$5 more than the former. What is the value of each horse?

39. Find three such numbers, that the first, with $\frac{1}{2}$ the sum of the second and third, shall be 78; the second, with $\frac{1}{3}$ the excess of the third over the first, shall be 60; and $\frac{1}{2}$ the sum of the three shall be 66.

40. If I had 3 shillings more in my pocket, I could give 2*s*. 6*d*. to each of a certain number of beggars; but if I give them only 2*s*. apiece, I shall have 4*s*. left. How much money have I in my pocket? What is the number of beggars?

41. A person had £27, 6*s*. in guineas and crown pieces. Having paid a debt of £14, 17*s*., he finds that he has as many guineas left, as he has paid away crowns; and as many crowns left, as he has paid away guineas. How many crowns and guineas had he at first?

Remark. A guinea is 21 shillings, and a crown 5 shillings, sterling.

42. A bill of £27, 9*s*. was paid in half-guineas and crowns; and twice the number of guineas was equal to half the number of crowns. How many of each were paid away?

43. A laborer agreed to work 24 days for 75 cents a day, and to forfeit his wages and 25 cents every day he was idle. At the end of the time, he received \$12. How many days was he idle?

44. A colonel would arrange a regiment of 1152 men in such a manner, that each rank may exceed

each file by 24 men. What numbers must he place in rank and file?

45. A cistern, containing 276 gallons, is emptied in 21 minutes by two cocks running successively. One cock discharges 16 gallons, and the other 11 gallons, in a minute. How many minutes is each cock running?

46. A merchant has two kinds of wine; one of which is worth 9*s*. 6*d*. per gallon, and the other, 13*s*. 6*d*. How many gallons of each must he take, to form a mixture of 104 gallons which shall be worth £56?

47. A gentleman bought a quantity of broadcloth for $48; and four times the number of yards were equal to three times the price of a yard. How many yards did he buy, and at what price?

48. Two gentlemen, A and B, have rectangular gardens contiguous to each other. A's garden is 20 yards wide, and $\frac{2}{3}$ as long as B's; and the surface of B's garden is to that of A's as 5 to 3. What is the width of B's garden?

49. A miser, dying, left a certain number of eagles, as many quarter-eagles, $\frac{3}{8}$ the number of half-eagles, and dollars enough to make the whole number of coins equal to $\frac{1}{5}$ of the value of the whole in dollars; and the eagles and dollars together were 2 more than $\frac{1}{2}$ the number of coins. How much money did he leave?

50. A farmer sold 120 bushels of rye and barley; receiving, for a bushel of each kind of grain, as many cents as there were bushels of that kind; and the barley brought only $\frac{4}{9}$ as much as the rye. How many bushels of each kind did he sell?

51. A gentleman distributed $47,50 among 30 men and women, giving the women 8*s*. and the men 10*s*. 6*d*. each. How many men and how many women were there?

52. A criminal, having escaped from prison, travelled 10 hours before his escape was known. He was then pursued, so as to be gained upon 3 miles an hour. After his pursuers had travelled 8 hours, they met an express going at the same rate as themselves, who met the criminal 2 hours and 24 minutes before. In what time from the commencement of the pursuit did they overtake him?

53. A farmer has an irregular piece of land, containing 5 acres, which he wishes to exchange for a square field of the same size. Required one of the sides of the square field.

Remark. An acre of land contains 160 square rods. Only an *approximate* answer to this question can be found, as the given quantity is not a perfect square.

54. I have a field containing 10 acres; and the length of the field exceeds its width by 18 rods. Required its dimensions.

55. A man bought a field whose length was to its breadth as 8 to 5. The number of dollars paid per acre was equal to the number of rods in the length of the field; and the number of dollars paid for the whole was equal to 13 times the number of rods round the field. What did he give for the field?

56. A father gave to each of his children, on new year's day, as many books as he had children; for each book he gave 12 times as many cents as there

were children; and the cost of the whole was \$15. How many children had he?

57. A messenger had been gone from a certain place 8 hours, when another was sent after him. The first went 7 miles an hour, and the second 11. In what time did the second overtake the first?

58. The members of a lyceum, wishing to buy an air-pump, found, if they paid 80 cents each, that they should raise \$18 more than they wanted for the purpose; but if they paid only 50 cents each, they would not have enough by \$12. What was the price of the air-pump?

59. A servant was sent to market with a basket of eggs, which he was directed to sell for 12 cents a dozen. Having carelessly broken 6 dozen of the eggs, he was obliged to get 15 cents a dozen for the rest, that there might be no loss. How many dozen of eggs did the basket contain at first?

60. A man wished to plant a certain number of trees in the form of a square. At the first trial, he had 39 trees left. He then determined to enlarge the square by adding one tree to each row; to do which, he found it necessary to procure 50 trees more How many trees had he at first?

61. A grocer, being asked the size of 3 wine-casks, replied, "If I fill the first empty cask from the second full cask, $\frac{4}{9}$ of the wine will remain; if I fill the second empty cask from the third full one, $\frac{1}{8}$ of the wine will remain; and the third empty cask will contain the contents of the first full cask and 23 gallons more." Required the size of the casks.

62. A cistern, which holds 2340 gallons, is filled in $\frac{3}{4}$ of an hour by 3 pipes; the first of which conveys 13 gallons more, and the second 6 gallons less, than the third per minute. How many gallons does each pipe convey in a minute?

63. Two persons, A and B, set out at the same time from two towns at the distance of 672 miles. B travelled 8 miles a day more than A; and when they had travelled half as many days as A went miles in a day, they met. How many miles did each travel daily?

64. A farmer has a rectangular peach-orchard, with unequal sides. If the difference of the sides be multiplied by the greater side, and the product divided by the less, the quotient is 24 rods; but if their difference be multiplied by the less side, and the product divided by the greater, the quotient is only 6 rods. What are the dimensions of the orchard?

65. There is a school-room in Boston, whose length is to its breadth as 6 to 5. If it were a square, having its sides equal to the length, it would contain 891 feet more than it would were the sides of the square equal to its width. What are the dimensions of the room?

66. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; but if the circumference of each wheel be increased one yard, it will make only 4 revolutions more in going the same distance. What is the circumference of each wheel?

67. When the passengers on board of a steam-boat

took their seats at the dinner-table, it was observed that the number of the men was to that of the women as 9 to 4; but when 6 men, with their wives, had retired, there remained at the table 3 times as many men as women. What was the number of passengers?

68. Two clerks, A and B, sent ventures in a ship bound to India. A gained \$11; and, at this rate, he would have gained as many dollars on a hundred as B sent out. B gained \$36, which was but one fourth part as much per cent. as A gained. How much money was sent out by each?

69. Required two fractions, whose product is $\frac{1}{6}$, and the sum of whose squares is $\frac{13}{36}$.

70. A company of persons spend £3 10*s*. at a tavern. Four of them go away without paying; in consequence of which, each of the others has to pay 2*s*. more than his proper share. How many persons were there in the company? and what was the proper share of each?

71. A gentleman bought a rectangular lot of land, giving \$10 for every foot in the perimeter. If the same quantity of land had been in the form of a square, and he had bought it in the same way, it would not have cost him so much by \$330; and if he had bought a square piece of the same perimeter, he would have had $12\frac{1}{4}$ rods more. What were the dimensions of the lot?

THE END.